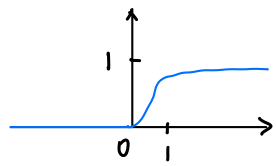
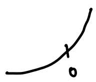
HW9

1. let $h(x) = \dfrac{e^{-\frac{1}{x^2}}}{e^{-\frac{1}{x^2}} + e^{-\frac{1}{(1-x)^2}}}$ $h(x)$:

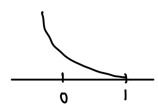


$f(x)$ is smooth since $g(x) = e^{-\frac{1}{x^2}}$:

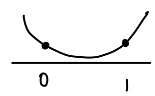


(numerator) is smooth ie. all derivatives $f^{(n)}$ exist.

$g(1-x) = e^{-\frac{1}{(1-x^2)}}$:



$g(x) + g(1-x)$:



(denominator) is strictly positive and smooth.

hence, $h(x) = \dfrac{e^{-\frac{1}{x^2}}}{e^{-\frac{1}{x^2}} + e^{-\frac{1}{(1-x)^2}}}$ is a smooth function with $f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x \geq 1 \\ \in [0,1] & x \in (0,1). \end{cases}$

2. let $g(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n+1} x^{n+1}$. Then $g(x)$ is differentiable everywhere in $\mathbb{R}$, ie. all derivatives $g^{(n)}$ exist in $\mathbb{R}$.

And $g'(x) = C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n$.

we have $g(0) = 0$ and $g(1) = C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$.

By MVT (Rolle), since $f(0) = f(1)$, then $\exists\, x \in (0,1)$ st. $f'(x) = 0$.

Hence $C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$ has a real root in $(0,1)$.

3. let $\delta > 0$, $|x-u| < \delta$ so that $|f'(x) - f'(u)| < \varepsilon$ for $\forall x \in [a,b]$.

Then if $0 < |t-x| < \delta$, by MVT (common), $\exists\, u \in [t,x]$ st. $f'(u) = \frac{f(t) - f(x)}{t-x}$.

Since $|x-u| = |u-x| < \delta$, so we have $|f'(u) - f'(x)| = \left| \frac{f(t)-f(x)}{t-x} - f'(x) \right| < \varepsilon$.

4. Differentiate $f(t) - f(\beta) = (t-\beta)Q(t)$ $n-1$ times at $t=\alpha$.

1st derivative $\quad f'(t) = 1 \cdot Q(t) + (t-\beta)Q'(t)$

nth deri. $\quad f^{(n)}(t) = nQ^{(n-1)}(t) + (t-\beta)Q^{(n)}(t) \qquad (*)$

(n-1)th deri. at $t=\alpha$: $f^{(n-1)}(\alpha) = (n-1)Q^{(n-2)}(\alpha) + (\alpha-\beta)Q^{(n-1)}(\alpha)$

from $(*)$ $\quad \dfrac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k = \dfrac{(\beta-\alpha)^k}{(k-1)!}Q^{(k-1)}(\alpha) + -\dfrac{(\beta-\alpha)^{k+1}}{k!}Q^{(k)}(\alpha)$

By Taylor's Thm, we have $f(\beta) - P_\alpha(\beta) = \dfrac{f^{(n)}(\gamma)}{n!}(\beta-\alpha)^n$

Define Taylor polynomial $\quad P_\alpha(\beta) = \displaystyle\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot (x-\alpha)^k$

$$= f(\beta) - \frac{f^{(n)}(\gamma)}{n!}(\beta-\alpha)^n$$

$$= f(\beta) - \frac{(\beta-\alpha)^n}{(n-1)!}Q^{(n-1)}(\alpha)$$

Hence, $f(\beta) = P_\alpha(\beta) + \dfrac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n$.

5. a) Assume f has 2 fixed points.

ie. $\exists x, y$ st. $f(x) = x$ and $f(y) = y$ where $x \neq y$.

By MVT (Common), $\exists z \in [x,y]$ st. $y - x = f(y) - f(x) = f'(z)(y-x)$.

So $f'(z) = 1$. Contradiction shows $x = y$, impling f has <u>at most</u> 1 fixed point.

b) By def of fixed point, ie. $f(t)$ has fixed point if $f(t) = t$.

Since $f(t) = t + \dfrac{1}{1+e^t}$.

In order for $f(t) = t$, $\dfrac{1}{1+e^t}$ need to be 0, which is impossible,

since $e^t > 0$, $\dfrac{1}{1+e^t} \neq 0$.

c) Suppose not. Assume f has no fixed point,

i.e. $\exists\, c \in (x_n, x_{n+1})$ s.t. $f(x_n) < x_n$ and $f(x_{n+1}) > x_{n+1}$.

By MVT, $\exists\, c \in (x_n, x_{n+1})$ s.t. $f'(c) = \left| \dfrac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} \right| \leq A > \left| \dfrac{x_{n+1} - x_n}{x_{n+1} - x_n} \right| = 1$

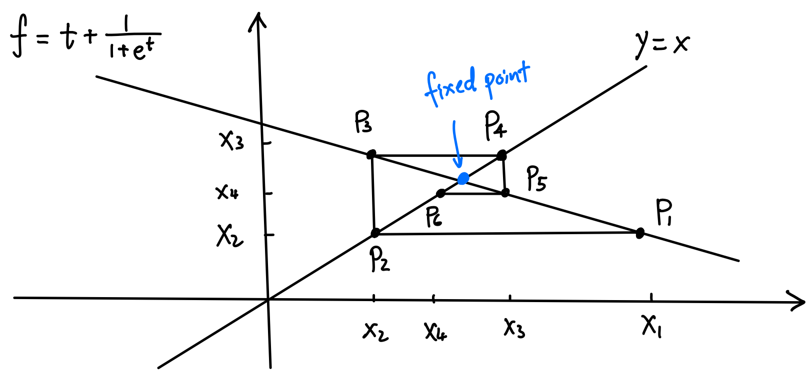
Contradiction shows f has a fixed point.

let x be the fixed point. Notice $\underset{x_n - x}{\left| f(x_{n+1}) - f(x_n) \right|} = \underset{\leq 1}{\underset{\uparrow}{f'(c)}} \cdot (x_{n+1} - x_n)$

$\therefore \lim\limits_{n\to\infty} \left| f(x_{n+1}) - f(x_n) \right| \longrightarrow 0$.

Hence $x = \lim(x_n)$.

d) $(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3 \to x_4) \to \cdots$



$x_{n+1} = f(x_n)$

$(x_1, f(x_1)) \to (x_n, y_n)$