HW4 Ross 12.10, 12.12, 14.2, 14.10
Rudin Chp3: 6, 7, 9, 11

12.10 Prove (S_n) is bounded iff $\limsup |S_n| < +\infty$.

"$\subseteq$" if (S_n) is bounded.

then there exists $M \in \mathbb{R}$ st. $|S_n| \leq M$ for $\forall n$.

$\therefore \sup\{|S_n| : n \in \mathbb{N}\} \leq M < +\infty \quad \forall n$.

hence $\limsup |S_n| < +\infty$.

"$\supseteq$" if $\limsup |S_n| < +\infty$.

then there exists $M \in \mathbb{R}$ st. $\limsup |S_n| = M$.

since sequence $A_N = \sup\{|S_n| : n > N\}$ converges to M, take $\varepsilon = 1$

then there exists $N_1 \in \mathbb{N}$ st. $\left| \sup\{|S_n| : n > N_1\} - M \right| < 1$

$\Rightarrow \quad \sup\{|S_n| : n > N_1\} < M+1$

$\Rightarrow \quad |S_n| < M+1 \quad \forall n > N$.

take $N = \max\{|S_1|, |S_2|, |S_3|, \dots, |S_N|, M+1\}$.

$\therefore |S_n| \leq N \quad \forall n$.

hence (S_n) is bounded.

$\square$

12.12 $\quad s_n$ sequence of nonnegative #s

$\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$

a) show $\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n$.

(hint: for last inequality, show first that $M > N$ implies $\sup\{\sigma_n : n > M\} \le \frac{1}{M}(s_1+s_2+\cdots+s_N) + \sup\{s_n : n > N\}$.

$\liminf \sigma_n \le \limsup \sigma_n$ is obvious. So only need to prove $\limsup \sigma_n \le \limsup s_n$, and $\liminf s_n \le \liminf \sigma_n$.

i) WTS $\quad \limsup \sigma_n \le \limsup s_n$

take M and N st. $n > M > N$.

then $\quad \sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$

$= \frac{1}{n}(s_1 + s_2 + \cdots + s_N + s_{N+1} + s_{N+2} + \cdots + s_n)$

$= \frac{1}{n}(s_1 + s_2 + \cdots + s_N) + \frac{1}{n}(s_{N+1} + s_{N+2} + \cdots + s_n)$

$\because n > M$

$\therefore \frac{1}{n}(s_1 + s_2 + \cdots + s_N) < \frac{1}{M}(s_1 + s_2 + \cdots + s_N)$

$\frac{1}{n}(s_{N+1} + s_{N+2} + \cdots + s_n) \le \sup\{s_n : n > M\}$

$\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_N) + \frac{1}{n}(s_{N+1} + s_{N+2} + \cdots + s_n)$

$< \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > M\}$

$\sup\{\sigma_n : n > M\} \le \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > M\}$

$\lim_{M\to\infty} \sup\{\sigma_n : n > M\} \le \lim_{M\to\infty} \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \lim_{M\to\infty} \sup\{s_n : n > M\}$

$\limsup \sigma_n \le 0 + \limsup s_n$

$\limsup \sigma_n \le \limsup s_n$.

ii) WTS $\quad \liminf s_n \le \liminf \sigma_n$

let $a_n = -\sigma_n$, $\quad b_n = -s_n$.

$\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$

$a_n = \frac{1}{n}(b_1 + b_2 + \cdots + b_n)$

$\because \limsup s_n = -\liminf(-s_n) \quad \forall s_n$

$\limsup(a_n) \le \limsup(b_n)$

$$-\limsup(a_n) \geq -\limsup(b_n)$$

$$-\limsup(-\sigma_n) \geq -\limsup(-S_n)$$

$$\liminf \sigma_n \geq \liminf S_n$$

$$\liminf S_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup S_n$$

$\square$

b) show if $\lim S_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim S_n$.

by Thm 10.7, if $\lim S_n$ exists $\Rightarrow \liminf S_n = \lim S_n = \limsup S_n$

by a) $\quad \liminf \sigma_n = \limsup \sigma_n$

(Thm 10.7) $\quad \lim \sigma_n = \liminf \sigma_n = \limsup \sigma_n = \lim S_n$

c) Give an example where $\lim \sigma_n$ exists, but $\lim S_n$ does not exist.

let $S_n = (1, 0, 1, 0, \cdots)$

$\lim S_n$ doesn't converge D.N.E.

$\lim \sigma_n \to \frac{1}{2}$

14.2 series converge ?

a) $\sum \frac{n-1}{n^2}$

$b_n = \frac{n}{2}$

$n-1 > \frac{n}{2}$ for $n \geq 10$

$\frac{n-1}{n^2} > \frac{\frac{n}{2}}{n^2}$ $\qquad \frac{n}{2} \cdot \frac{1}{n^2} = \frac{1}{2n}$

$\frac{n-1}{n^2} > \frac{1}{2n}$

$P \leq 1$ Div. by P-series test

Div. by Direct Comparsion test (DCT)

b) $\sum (-1)^n$

$\lim_{n\to\infty} (-1)^n \not\to 0$

$\therefore$ diverge by test for Divergence (TFD).

c) $\sum \frac{3n}{n^3}$

$= \sum \frac{3}{n^2} = 3\sum \frac{1}{n^2} \rightarrow P>1$

$\therefore$ converge by P-series

d) $\sum \frac{n^3}{3^n}$

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{\frac{(n+1)^3}{3^{(n+1)}}}{\frac{n^3}{3^n}} \right| = \lim \left| \frac{(n+1)^3}{3^{(n+1)}} \cdot \frac{3^n}{n^3} \right|$

$= \lim \left| \frac{(n+1)^3}{3n^3} \right| = \lim_{n\to\infty} \left| \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \right|$

$= \frac{1}{3} < 1$

$\therefore$ converge absolutely by Ratio test

e) $\sum \frac{n^2}{n!}$

$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(n+1)\cdot n^2} \right|$

$= \lim_{n\to\infty} \left| \frac{n+1}{n^2} \right| = \lim \left| \frac{1}{n} + \frac{1}{n^2} \right| = 0$

< 1

$\therefore$ converge absolutely by Ratio test

f) $\sum \frac{1}{n^n}$

$r_n = \sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{1}{n^n}\right|} = \frac{1}{n}$

$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim \frac{1}{n} = 0$

< 1

$\therefore$ conv. absolutely by Root test

g) $\sum \frac{n}{2^n}$

$= \lim_{n\to\infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{2n} \right|$

$= \lim_{n\to\infty} \left| \frac{1}{2}\left(1+\frac{1}{n}\right) \right| = \frac{1}{2} < 1$

$\therefore$ conv. abso. by Ratio test.

Rudin Chp3 : 6, 7, 9, 11

6. convergence / divergence ?

a) $a_n = \sqrt{n+1} - \sqrt{n}$

partial sum $S_n = \sum_{j=1}^{n} a_n = \sqrt{n+1} - 1$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\sqrt{n+1} - 1) = \infty$

$\therefore$ Diverges

b) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$

$= \dfrac{n+1-n}{n(\sqrt{n+1}+\sqrt{n})} < \dfrac{1}{n^{3/2}}$ $\quad p = 3/2 > 1$

$\uparrow$ conv. by P-series

$\therefore$ div. by comparison test

c) $a_n = (\sqrt[n]{n} - 1)^n$

$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} (\sqrt[n]{n} - 1) = n^{\frac{1}{n}} - 1 = n^0 - 1 = 1 - 1 = 0 < 1$

$\therefore$ conv. by Root test