Ex3) Need $f^n(0)=0 \;\forall n \in \{0,1,2,\ldots\}$

Need $f^n(1)=0 \;\forall n \in \{1,2,\ldots\}$

$\lim_{x\to 1} f(x)=1$

~~If $f=g\cdot h$ then $f'=g'h+gh'$~~

~~But if both $g(1)\neq 0$ & $h(1)\neq 0$ then for $f'(1)=0$, $g'h=-gh'$~~

OK so let's look at $g(x)=e^{-\frac{1}{x}}e^{\frac{1}{x-1}}$ which builds on example 3 to still be smooth and keeps condition $g^n(0)=0 \;\forall n\in\{0,1,\ldots\}$ and $g^n(1)=0 \;\forall n\in\{1,2,\ldots\}$

However, $g(1)=0$. But note that for $x\in(0,1)$, $g(x)>0$.

What if we take the integral of $g(x)$. First of all,

$$g(x)=e^{\frac{1}{x-1}-\frac{1}{x}}=e^{\frac{x}{x(x-1)}-\frac{x-1}{x(x-1)}}=e^{\frac{1}{x(x-1)}}$$

If $f(x)=\int_0^x g(t)\,dt$

then $f^n(x)=g^{(n-1)}(x)$ by fundamental thm of calculus.

Since $g^n(x)=0 \;\forall n\in\{0,1,2,\ldots\}$

$f^n(x)=0 \;\forall n\in\{1,2,\ldots\}$ ✓

suffice to check that

$f(0)=\int_0^0 g(t)\,dt=0$ ✓

$\lim_{x\to 1^-} f(x)=c$, which $c\neq 1$, but since $c\neq 0$, c finite

we divide f by c... $c=\int_0^1 e^{\frac{1}{t-1}-\frac{1}{t}}\,dt$

So we construct

$$\boxed{h(x)=\begin{cases} 0 & x\leq 0 \\ \dfrac{\int_0^x e^{\left(\frac{1}{t-1}-\frac{1}{t}\right)}dt}{\int_0^1 e^{\left(\frac{1}{t-1}-\frac{1}{t}\right)}dt} & \\ 1 & x\geq 1 \end{cases}}$$

Ex4) $f(x) = C_0 x + \frac{c_1}{2}x^2 + \dots \frac{c_{n-1}}{n}x^n + \frac{c_n}{n+1}x^{n+1}$ polynomial, it is continuous and infinitely differentiable.

$$f'(x) = C_0 + C_1 x + \dots C_{n-1}x^{n-1} + C_n x^n$$

We know $f(0)=0$, since $x=0 \Rightarrow \frac{c_{i-1}}{i}x^i = 0$ if $i \geq 1$

But also, it is given that $f(1)=0$.

So by rolle's thm, $f'(x)=0$ for some $x \in (0,1)$

so there's a root to $C_0 + C_1 x + \dots + C_{n-1}x^{n-1} + C_n x^n$ between 0, 1.

Ex8) $f'(x) = \lim_{a \to x} \frac{f(a)-f(x)}{a-x}$. Since f' is continuous,

$\lim_{b \to x} f'(x) = f'(x)$ on $[a,b]$

Take δ such that $\forall z \in (x-\delta, x+\delta)\ |f'(z) - f'(x)| < \varepsilon$

for whatever t you want with $|t-x|<\delta$ → Fix t such that $|t-x|<\delta$

By MVT, there exists $j \in (x,t)$ st

$$f'(j) = \frac{f(t)-f(x)}{t-x}$$

Since $j \in (x-\delta, x+\delta)$, $\boxed{\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \varepsilon}$

Ex 18) $F^{n-1}(\alpha) = -\beta Q^{n-1}(\alpha) + \alpha Q^{n-1}(\alpha) + n_1 Q^{n-1}(\alpha)$.

$h + th'$

$h' + h' + th''$

$h'' + h'' + h'' + th''$

22. a) Since f is differentiable, either $f'(t)>1$ or $f'(t)<1$ $\forall t$.

If $f(p)=p$, and $F(K)=K$ and $p \leq K$ (WLOG)
then by Rolle's thm, $\exists$ $c \in [p, K]$
such that $f'(c) = \frac{p-K}{p-K} = 1$ which violates assumption so $p=K$. At most one

b) Since $e^t>0$, $1+e^t>0$, $(1+e^t)^{-1}>0$, $(1+e^t)^{-1}+t>t$
so no fixed points since $f(t)>t$, $\forall t$.

$$0 < f'(t) = 1 + \frac{-e^t}{(1+e^t)^2} < 1$$