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hw2

Ross 9.9
(a)
Let $M > 0$. Let $N_* = \max\{N_0, N\}$ where $N \in \mathbb{N}$ is the number such that $s_n > M \quad \forall n > N$. (note the existence of N is guaranteed by definition 9.8). Then $\forall n > N_*$ we have $M < s_n \le t_n$, i.e $M < t_n$. Hence $\lim t_n = \infty$ by definition.
(b) Similar to (a). Let $M < 0$. Let $N_* = \max\{N_0, N\}$ where $N \in \mathbb{N}$ is the number such that $t_n < M \; \forall n > N$. Then for $n > N_*$, we have $s_n \le t_n < M$, i.e $s_n < M$. Hence $\lim s_n = -\infty$ by definition.
(c)
Let $L_s = \lim s_n$, $L_t = \lim t_n$. Suppose $L_s > L_t$.
Let $L_0 = \frac{L_s + L_t}{2}$, and $\delta = L_s - L_t > 0$.
By definition of limit $\exists N_s, N_t \in \mathbb{N}$ such that $|s_n - L_s| < \frac{\delta}{2}$ for $n > N_s$ and $|t_n - L_t| < \frac{\delta}{2}$ for $n > N_t$.
Let $N = \max\{N_s, N_t\}$.
Then for $n > N$, we have $t_n < \frac{\delta}{2} + L_t = L_0 = L_s - \frac{\delta}{2} < s_n$.
Hence for $n > N$, we have $t_n < s_n$ and $s_n \le t_n$ simultaneously. This is clearly nonsense, hence the assumption that $L_s > L_t$ is impossible. So $L_s \le L_t$. QED.

Ross 9.15

Suppse $a \neq 0$(if $a = 0$ then $(0^n/n!) = 0 \ \forall n$).

Define $\phi(a) =$ the smallest natural number such that $\frac{|a|}{n} < 1 \ \forall n > \phi(a)$. As an example, $\phi(3) = 4, \phi(2.1) = 3$.

Then

$$\begin{aligned} |\frac{a^n}{n!} - 0| &= \frac{|a|^n}{1 \cdot 2 \cdot ... \cdot \phi(a) \cdot ... \cdot n} \\ &= \frac{|a|^{\phi(a)}}{1 \cdot 2 \cdot ... \cdot \phi(a)} \times \frac{|a|^{n-\phi(a)}}{(\phi(a)+1) \cdot ... \cdot n} \end{aligned}$$

Looking at the second fraction in the previous line , there are a total of $n - \phi(a)$ terms in the numerator, and the same number of terms on the denominator *(you may wonder what if $\phi(a) > n$. This is not an issue because at the end we can simply require n to be at least bigger than $\phi(a)$)* . So it can be written as a product of fractions of the form $\frac{|a|}{m}$, where each integer $m \geq \phi(a) + 1 > \phi(a)$. By the definition of ϕ, such fractions are bounded above by 1. For convenience, take only the last fraction in this product(i.e $\frac{|a|}{n}$) and be aware that the whole product is no greater than this last term. So now going back up to what we had before, $|\frac{a^n}{n!}| \leq \frac{|a|^{\phi(a)}}{1 \cdot 2 \cdot ... \cdot \phi(a)} \cdot \frac{|a|}{n}$. Notice the first factor on the right hand side does not depend on n. So we call it C. If we require $C \cdot \frac{|a|}{n} < \epsilon$ where $\epsilon > 0$, then $n > \frac{C|a|}{\epsilon}$. Thus $\forall \epsilon > 0$, if $n > \max\{\frac{C|a|}{\epsilon}, \phi(a)\}$ then $|\frac{a^n}{n!} - 0| < \epsilon$. Therefore $\lim \frac{a^n}{n!} = 0$.

(Note that ϕ is also known as the floor function, but plus 1.)

Ross 10.7

Let $A = \sup S$.
Let $\epsilon > 0$, and suppose $A - \epsilon$ is an upper bound of S.
Then $A \leq A - \epsilon$ by definition of sup, which is impossible.
Hence there does not exist $\epsilon > 0$ such that $A - \epsilon$ is an upper bound of S.
Therefore $\forall \epsilon > 0$, $\exists s \in S$, call it s_*, such that $s_* > A - \epsilon$ (because otherwise $A - \epsilon$ would be an upper bound of S). $(*)$
So let $\epsilon > 0$ and denote the corresponding s_* as s_1. Then let $\epsilon = \frac{A - s_1}{2} > 0$, and denote the corresponding s_* as s_2. Similarly, there are s_3, s_4, ..., and the sequence they form converges to A. The proof that it converges to A follows trivially from $(*)$ ($s_* > A - \epsilon \implies |A - s_*| = A - s_* < \epsilon$).

Ross 10.8

Let $n \in \{1, 2, 3, ...\}$.

$$\begin{aligned}
\sigma_{n+1} - \sigma_n &= \frac{(s_1 + ... + s_n) + s_{n+1}}{n+1} - \frac{s_1 + ... + s_n}{n} \\
&= \frac{n(s_1 + ... + s_n) + ns_{n+1} - (n+1)(s_1 + ... + s_n)}{n(n+1)} \\
&= \frac{ns_{n+1} - (s_1 + ... + s_n)}{n(n+1)} \\
&= \frac{(s_{n+1} - s_1) + (s_{n+1} - s_2) + ... + (s_{n+1} - s_n)}{n(n+1)} \geq 0 \text{ (recall } s_n \text{ is increasing)}
\end{aligned}$$

Since n was arbitrary, by definition (σ_n) is increasing.

Ross 10.9

(a)$s_2 = \frac{1}{2}, s_3 = \frac{2}{3}\frac{1}{4} = \frac{1}{6}, s_4 = \frac{3}{4}\frac{1}{36} = \frac{1}{48}$

(b) From looking at part (a) it's obvious $0 < s_n \leq 1\ \forall n \in \mathbb{N}$. So $s_n^2 \leq s_n\ \forall n$. Also $0 \leq \frac{n}{n+1} < 1\ \forall n \in \mathbb{N}$. Thus $s_{n+1} = \frac{n}{n+1}s_n^2 < 1 \cdot s_n^2 < s_n$. So (s_n) is bounded and decreasing, and by theorem 10.2 it converges.

(c) $\lim s_n = \lim s_{n+1} = \lim \frac{n}{n+1} \lim s_n^2 = 1 \cdot \lim s_n^2 \implies \lim s_n = 1$ or 0. It's obviously not 1, because it's decreasing and s_1 is already 1. Therefore $\lim s_n = 0$.

Ross 10.10

(a) $s_1 = 1, s_2 = \frac{2}{3}, s_3 = \frac{5}{9}, s_4 = \frac{14}{27}$.

(b) Suppose that $s_n > \frac{1}{2}$, for some n. Then $s_{n+1} = \frac{s_n+1}{3} > \frac{\frac{1}{2}+1}{3} = \frac{3}{6} = \frac{1}{2}$. Also $s_1 = 1 > \frac{1}{2}$. Therefore $s_n > \frac{1}{2} \forall n \in \{1, 2, 3, ...\}$.

(c)

$$\begin{aligned} s_{n+1} - s_n &= \frac{1+s_n}{3} - s_n \\ &= \frac{1+s_n-3s_n}{3} = \frac{1-2s_n}{3} \end{aligned}$$

Hence

$$\begin{aligned} 3(s_{n+1} - s_n) &= 1 - 2s_n \\ 3s_{n+1} &= 1 + s_n \\ \frac{s_{n+1}}{s_n} &= \frac{1}{3}(1 + \frac{1}{s_n}) \text{ (recall } s_n > \frac{1}{2} \text{ so } \frac{1}{s_n} < 2 \text{ for all n)} \\ &< \frac{1}{3}(1+2) = 1 \\ s_{n+1} &< s_n \end{aligned}$$

(d)Since (s_n) is a decreasing sequence, it is monotone and $s_n \leq s_1 \ \forall n$. Also recall $s_n > \frac{1}{2} \ \forall n$. Therefore it is bounded and monotone, and by theorem 10.2 it converges.

Let $L = \lim s_n$. Then $\lim s_{n+1} = L$ as well (note that (s_{n+1}) is a subsequence of (s_n) and theorem 11.3 can be used here). Then $\lim s_{n+1} = \lim \frac{s_n+1}{3} = \frac{1}{3}(\lim s_n + \lim 1) \implies L = \frac{1}{3}(L+1) \implies L = \frac{1}{2}$.

Ross 10.11

(a) Suppose at some $n \in \mathbb{N}$, $t_{n+1} < 0$. Then $(1 - \frac{1}{4n^2})t_n < 0 \implies t_n < 0$ (since the term in parenthesis is obviously nonnegative). So if one term is negative, every previous term in this sequence is also negative, which is clearly false. Therefore the aforementioned number n does not exist; i.e $t_{n+1} >= 0 \; \forall n \in \{1, 2, 3, ...\}$.
Now suppose at some $n \in \mathbb{N}$, $t_{n+1} > 1$. Then $(1-\frac{1}{4n^2})t_n > 1 \implies t_n > \frac{1}{1-\frac{1}{4n^2}} = \frac{4n^2}{4n^2-1} > \frac{4n^2}{4n^2} = 1$. So if one term is greater than 1, every previous term is greater than 1. This is clearly nonsense($s_1 = 1$, for example), therefore the aforementioned number n does not exist; i.e $t_{n+1} \leq 1 \forall n \in \{1, 2, 3...\}$.
So we now see that $0 \leq t_n \leq 1$, meaning the sequence is bounded.
$\frac{t_{n+1}}{t_n} = 1 - \frac{1}{4n^2} < 1 \; \forall n \in \{1, 2, 3...\}$. So $t_{n+1} < t_n$, which means this sequence is decreasing.
Since this sequence is both bounded and decreasing, by theorem 10.2 it converges.
(b) $\lim t_n = \lim_{n->\infty} \left(\frac{4n^2-1}{4n^2} \cdot \frac{4(n-1)^2-1}{4(n-1)^2} \cdot ... \cdot \frac{3}{4} \right)$. I'm unable to figure out an exact number for this, but I will give a rough approximation.
Define the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(x + 1) = (1 - \frac{1}{4x^2})f(x)$. Then expanding the left hand side up to first order in 1 we have

$$f(x + 1) = f(x) + 1 \cdot f'(x) + O(f''(x)) \approx f(x) + f'(x)$$

So

$$\begin{aligned} f(x) + f'(x) &\approx (1 - \frac{1}{4x^2})f(x) \\ f'(x) &\approx -\frac{1}{4x^2} f(x) \\ \frac{df}{f} &\approx -\frac{1}{4x^2} dx \\ \int \frac{df}{f} &\approx -\int \frac{1}{4x^2} dx \\ \ln f &\approx \frac{1}{4x} + C \\ f &\approx Ce^{\frac{1}{4x}} \end{aligned}$$

Plugging in the initial condition that $f(1) = \frac{3}{4}$, we have

$$\frac{3}{4} = Ce^{\frac{1}{4}} \implies C = \lim_{x->\infty} f(x) = \frac{3/4}{e^{1/4}} \approx 0.6$$

This approximation appears to be within 7% of the true result.

2. Squeeze theorem

Define $\mu_n = b_n - a_n$, $\nu_n = c_n - b_n$.

$b_n \geq a_n \forall n \implies \mu_n \geq 0$. Similarly, $\nu_n \geq 0$.

$\mu_n + \nu_n = b_n - a_n + c_n - b_n = c_n - a_n$

$\implies \lim(\mu_n + \nu_n) = \lim(c_n - a_n) = \lim c_n - \lim a_n = L - L = 0$

$\implies \lim \mu_n + \lim \nu_n = 0$

$\implies \lim \mu_n = -\lim \nu_n \qquad (*)$

Also, since $\mu_n \geq 0, \nu_n \geq 0 \forall n$, it must be that $\lim \mu_n \geq 0$ and $\lim \nu_n \geq 0$. This, combined with $(*)$, implies that $\lim \mu_n = \lim \nu_n = 0$.

$\lim(b_n - L) = \lim(c_n - \nu_n - L) = \lim(c_n - L) - \lim \nu_n = 0 - 0 = 0$, hence $\lim b_n = L$. QED