1.10

Our n^{th} proposition is

$(2n+1)+(2n+3)+\cdots+(4n-1)=3n^2$

We proceed by mathematical induction.

for $n=1$, $2(1)+1=3=3(1)^2$

Now assume the proposition holds.

for $n=k+1$

$(2(k+1)+1)+(2(k+1)+3)+\cdots+(4(k+1)-1$

$=(2k+3)+(2k+5)+\cdots+(4k-1)+(4k+1)+4k+3$

$=3k^2-2k-1+4k+1+4k+3$

$=3k^2+6k+3$

$=3(k^2+2k+1)$

$=3(k+1)^2$

by the principle of mathematical induction,

P_n is true for all n. $\square$

1.12

(a) $n=1$, $(a+b)^1 = \binom{1}{0}a^1 + \binom{1}{1}a^0 b$

$= \frac{1}{0!(1-0)!}a^1 + \frac{1!}{1!(1-1)!}b$

$= a+b$

for $n=2$, $(a+b)^2 = a^2+2ab+b^2$

$(a+b)^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2$

$= \frac{2!}{0!(2)!}a^2 + \frac{2!}{1!(1)!}ab + \frac{2!}{2!(0)!}b^2$

$= a^2 + 2ab + b^2$

for $n=3$, $(a+b)^3 = (a^2+2ab+b^2)(a+b)$

$= a^3+3a^2b+3ab^2+b^3$

$(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$

$= \frac{3!}{0!\,3!}a^3 + \frac{3!}{1!\,2!}a^2b + \frac{3!}{2!\,1!}ab^2 + \frac{3!}{3!\,0!}b^3$

$= a^3+3a^2b+3ab^2+b^3$

(b) $\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$

$= n!\left(\frac{n-k+1}{k!(n-k+1)!} + \frac{k}{k!(n-k+1)!}\right) = n!\left(\frac{n+1}{k!(n+1-k)!}\right)$

$= \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$

(c) For $n=1$, $(a+b) = \binom{1}{0} a + \binom{1}{1} b$

Now suppose this is true for some $n \in \mathbb{N}$

For $n+1$,

$(a+b)^{n+1} = (a+b)(a+b)^n$

$= (a+b)\left[\binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{n} b^n\right]$

$= a\left[\binom{n}{0} a^n + \cdots + \binom{n}{n} b^n\right] + b\left[\binom{n}{0} a^n + \cdots + \binom{n}{n} b^n\right]$

$= \binom{n}{0} a^{n+1} + \binom{n}{1} a^n b + \cdots + \binom{n}{n} a b^n + \binom{n}{0} a^n b + \cdots + \binom{n}{n} b^{n+1}$

$= \binom{n}{0} a^{n+1} + \left[\binom{n}{1} + \binom{n}{0}\right] a^n b + \left[\binom{n}{2} + \binom{n}{1}\right] a^{n-1} b^2 + \cdots + \binom{n}{n} b^{n+1}$

$= \binom{n}{0} a^{n+1} + \binom{n+1}{1} a^n b + \cdots + \binom{n+1}{n} a b^n + \binom{n}{n} b^{n+1}$

$= \binom{n+1}{0} a^{n+1} + \binom{n+1}{1} a^n b + \cdots + \binom{n+1}{n} a b^n + \binom{n+1}{n+1} b^{n+1}$ ▨

2.1

$x^2-3=0$, Possible rational solutions are $\pm 1, \pm 3$

none of this satisfies, $\sqrt{3}$ is not rational.

$x^2-5=0$, $\pm 1, \pm 5$ are the only possible rational solutions,

$\sqrt{5}$ is not rational

$x^2-7=0$, $\pm 1, \pm 7$ are the only possible rational solutions

$\sqrt{7}$ is not rational.

$x^2-24=0$. $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$ are the

only Possible rational solutions, but none of these

satisfy $\therefore \sqrt{24}$ is not rational.

$x^2-31=0$. $\pm 1, \pm 31$ are the only possible rational

solutions, none of these satisfy.

$\therefore \sqrt{31}$ is not rational.

2.2

$x^3-2=0$, $\pm 1, \pm 2$ are the only possible Solutions. But none of these Satisfy
$\therefore \sqrt[3]{2}$ is not rational

$x^7-5=0$, $\pm 1, \pm 5$ are the only possible rational solutions. But none of these satisfy. $\therefore \sqrt[7]{5}$ is not rational.

$x^4-13=0$, $\pm 1, \pm 13$ are the only possible rational solutions. But none of these satisfy.
$\therefore \sqrt[4]{13}$ is not rational.

2.7

$$x = \sqrt{4+2\sqrt{3}} - \sqrt{3}$$

$$= \sqrt{1+2\sqrt{3}+3} - \sqrt{3}$$

$$= \sqrt{(1+\sqrt{3})^2} - \sqrt{3}$$

$$= 1$$

(b) $x = \sqrt{6+4\sqrt{2}} - \sqrt{2}$

$$= \sqrt{4+4\sqrt{2}+2} - \sqrt{2}$$

$$= \sqrt{(2+\sqrt{2})^2} - \sqrt{2}$$

$$= 2+\sqrt{2} - \sqrt{2}$$

$$= 2$$

3.6

(a) $|a+b+c| \le |a+b| + |c|$

Since $|a+b| \le |a| + |b|$

$|a+b+c| \le |a| + |b| + |c|$

(b) $|a_1| \le |a_1|$

We assume $|a_1 + a_2 + \cdots + a_n| \le |a_1| + \cdots + |a_n|$

$|a_1 + a_2 + \cdots + a_{n+1}| \le |a_1 + a_2 + \cdots + a_n| + |a_{n+1}|$

by triangle inequality,

by the assumption, $|a_1 + \cdots + a_{n+1}| \le |a_1| + \cdots + |a_{n+1}|$.

Thus by the principle of mathematical induction,

$|a_1 + a_2 + \cdots + a_{n+1}| \le |a_1| + |a_2| + \cdots + |a_{n+1}|$. $\blacksquare$

4.11

By the denseness of $\mathbb{Q}$, $\exists\, r_1 \in \mathbb{Q}$ s.t. $a < r_1 < b$.

Since $r_1, b \in \mathbb{R}$, $\exists\, r_2 \in \mathbb{Q}$ s.t. $a < r_1 < r_2 < b$ by the denseness of $\mathbb{Q}$. We can recursively define rational numbers between r_{n-1} and b in the same manner by the denseness of $\mathbb{Q}$, then one can show there exist infinitely many rationals between a and b.

4.14

(a) Prove $\sup(A+B) = \sup(A) + \sup(B)$

For $a \in A$ and $b \in B$, $(a+b) \in A+B$.

$a+b \leq \sup(A+B)$.

$a \leq \sup(A+B) - b$, this shows $\sup(A+B) - b$ is an least upper bound of A. This gives us,

$\sup(A) \leq \sup(A+B) - b$ and $b \leq \sup(A+B) - \sup(A)$

this shows b is bounded above by $\sup(A+B) - \sup(A)$

$\therefore \sup(B) \leq \sup(A+B) - \sup(A)$

$\sup(A) + \sup(B) \leq \sup(A+B)$

$\Rightarrow \sup(A+B) = \sup(A) + \sup(B)$

(b) Prove $\inf(A+B) = \inf(A) + \inf(B)$.

We have $\inf(S) = -\sup(-S)$

For $(a+b) \in A+B$, we have

$a+b \geq \inf(A+B)$.

$a \geq \inf(A+B) - b$

$\Rightarrow \inf(A) \geq \inf(A+B) - b$.

$\Rightarrow b \geq \inf(A+B) - \inf(A)$.

$\inf(B) \geq \inf(A+B) - \inf(A)$

$\inf(A) + \inf(B) \geq \inf(A+B)$

$\inf(A+B) = \inf(A) + \inf(B)$.

7.5

(a) $S_n = \sqrt{n^2+1} - n \cdot \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n}$

$\lim \frac{1}{\sqrt{n^2+1}+n} = 0$.

(b) $(\sqrt{n^2+n} - n)\cdot(\sqrt{n^2+n}+n) = n^2+n-n^2$

$\Rightarrow \sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\frac{\sqrt{n^2+n}}{n}+1} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$

$\lim \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$

(c) $(\sqrt{4n^2+n} - 2n)\cdot(\sqrt{4n^2+n}+2n) = 4n^2+n-4n^2 = n$.

$\sqrt{4n^2+n} - 2n = \frac{n}{\sqrt{4n^2+n}+2n} = \frac{1}{\sqrt{4+\frac{1}{n}}+2}$

$\lim \frac{1}{\sqrt{4+\frac{1}{n}}+2} = \frac{1}{4}$.