MATH 104 HW #1

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Problem 1. *Ross 1.10*

Prove $(2n+1)+(2n+3)+(2n+5)+\cdots+(4n-1)=3n^2$ *for all positive integers* n*.*

Proof. For simplicity, let us rewrite the summation as $\sum_{k=1}^{n} 2(n+k)-1$. We shall prove this using induction. The base case ($n=1$) is easy to verify. Assume that the above holds for n. Then,

$$\begin{aligned}\sum_{k=1}^{n+1} 2((n+1)+k)-1 &= \sum_{k=1}^{n+1} 2+\sum_{k=1}^{n+1} 2(n+k)-1 \\ &= 2(n+1)+2(n+(n+1))-1+\sum_{k=1}^{n} 2(n+k)-1 \\ &= 6n+3+3n^2 = 3(n+1)^2\end{aligned}$$

Thus, we have proved that the above must hold for all positive integers n. □

Problem 2. *Ross 1.12*

For $n \in \mathbb{N}$*, let* $n!$ *denote the product* $1 \cdot 2 \cdot 3 \cdots n$*. Also let* $0! = 1$ *and define*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ for } k = 0, 1, \cdots, n.$$

The binomial theorem asserts that

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

a) Verify the binomial theorem for $n = 1, 2,$ *and* 3*.*
b) Show $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ *for* $k = 1, 2, \cdots, n$*.*
c) Prove the binomial theorem using mathematical induction and part b).

Proof. First, we verify the binomial theorem for $n = 1, 2,$ and 3.

- $(a+b)^1 = a+b = \binom{1}{0}a + \binom{1}{1}b$
- $(a+b)^2 = a^2+2ab+b^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2$
- $(a+b)^3 = a^3+3a^2b+3ab^2+b^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$

Next, we demonstrate a few short properties which can help us prove the binomial theorem.

$$\begin{aligned}\binom{n}{k}+\binom{n}{k-1} &= \frac{n!}{k!(n-k)!}+\frac{n!}{(k-1)!(n+1-k)!}\\ &= (n+1-k)\frac{n!}{k!(n+1-k)!}+k\frac{n!}{k!(n+1-k)!}\\ &= (n+1)\frac{n!}{k!(n+1-k)!}=\frac{(n+1)!}{k!(n+1-k)!}\\ &= \binom{n+1}{k} \text{ for } k=1,2,\cdots,n\\ \binom{n}{0}=\binom{n}{n} &= 1 \text{ for } n\in\mathbb{N}\end{aligned}$$

For simplicity, we will rewrite the binomial theorem in summation form:

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

We will prove the binomial theorem using induction. We have already shown above the base case ($n=1$) is true. Assume that the binomial theorem holds for n. Then,

$$\begin{aligned}(a+b)^{n+1} &= (a+b)(a+b)^n=(a+b)\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\\ &= \sum_{k=0}^{n}\binom{n}{k}a^{n+1-k}b^k+\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}\\ &= \binom{n}{0}a^{n+1}+\binom{n}{n}b^{n+1}+\sum_{k=1}^{n}\binom{n}{k}a^{n+1-k}b^k+\sum_{k=0}^{n-1}\binom{n}{k}a^{n-k}b^{k+1}\\ &= \binom{n}{0}a^{n+1}+\binom{n}{n}b^{n+1}+\sum_{k=1}^{n}\binom{n}{k}a^{n+1-k}b^k+\sum_{k=1}^{n}\binom{n}{k-1}a^{n+1-k}b^k\\ &= \binom{n+1}{0}a^{n+1}+\sum_{k=1}^{n}\binom{n+1}{k}a^{n+1-k}b^k+\binom{n+1}{n+1}b^{n+1}\\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}a^{n+1-k}b^k.\end{aligned}$$

Thus, by induction, the binomial theorem must be true for all $n\in\mathbb{N}$. □

Theorem 1. *For $r,n\in\mathbb{N}$, $\sqrt[r]{n}$ is rational if and only if there exists $q\in N$ among the divisors of n such that $q^r=n$.*

Proof. Let us first prove the forward direction. Regardless if $\sqrt[r]{n}$ is rational, $\sqrt[r]{n}$ is a solution to the polynomial equation $x^r - n = 0$. By the Rational Root Theorem, if a rational solution to the equation exists, it must be a divisor of n. Letting this divisor be q, we have $q^r - n = 0 \Rightarrow q^r = n$.

The reverse direction is more straightforward. If there exists a q such that $q^r = n$, then $\sqrt[r]{n} = \sqrt[r]{q^r} = q$ which is rational. □

Problem 3. *Ross 2.1*

Show $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{24}$, and $\sqrt{31}$ are not rational numbers.

Proof. The square of the divisors of 3, 5, 7, 24, and 31 do not equal 3, 5, 7, 24, and 31, respectively. Thus, by Theorem 1, they cannot be rational numbers. □

Problem 4. *Ross 2.2*

Show $\sqrt[3]{2}, \sqrt[7]{5}$, and $\sqrt[4]{13}$ are not rational numbers.

Proof. 2, 5, and 13 are prime numbers. Becuase their only divisors are 1 and themselves, the nth power of their divisors cannot equal themselves for $n \neq 1$. Thus, by Theorem 1, they cannot be rational numbers. □

Problem 5. *Ross 2.7*

Show the following irrational-looking expressions are actually rational numbers: (a) $\sqrt{4+2\sqrt{3}} - \sqrt{3}$, and (b) $\sqrt{6+4\sqrt{2}} - \sqrt{2}$.

- $\sqrt{4+2\sqrt{3}} - \sqrt{3} = \sqrt{(1+\sqrt{3})^2} - \sqrt{3} = 1$
- $\sqrt{6+4\sqrt{2}} - \sqrt{2} = \sqrt{(2+\sqrt{2})^2} - \sqrt{2} = 2$

Problem 6. *Ross 3.6*

(a) Prove $|a+b+c| \leq |a|+|b|+|c|$ for all $a, b, c \in \mathbb{R}$.

(b) Use induction to prove

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

for n numbers $a_1, a_2, \cdots, a_n$.

Proof. We first prove the Triangle Inequality for 3 numbers.

$$|a+b+c| = |(a+b)+c| \leq |a+b| + |c| \leq |a+b+c|.$$

Now, let us prove the Triangle Inequality generally for any amount of numbers using induction. The trivial case ($n = 1$) and the base case ($n = 2$) is true. Suppose that the Triangle Inequality holds for n numbers. Then, we have

$$\begin{aligned} |a_1 + a_2 + \cdots + a_n + a_{n+1}| &= |(a_1 + a_2 + \cdots + a_n) + a_{n+1}| \\ &= |a_1 + a_2 + \cdots + a_n| + |a_{n+1}| \\ &= |a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}|. \end{aligned}$$

Thus, the Triangle Inequality holds for all $n \in \mathbb{N}$. □

Problem 7. *Ross 4.11*

Consider $a, b \in \mathbb{R}$ where $a < b$. Use Denseness of $\mathbb{Q}$ to show that there are infinitely many rationals between a and b.

Proof. Suppose that the number of rationals between a and b is finite. Call this set of rationals S. Because $S \subset \mathbb{R}$ and is finite, $m = \max(S)$ must exist. Because $m, b \in \mathbb{R}$, by the Denseness of $\mathbb{Q}$ there must exist a rational number $a < m < r < b$. Because r is rational, we must have $r \in S$. However, because $r > m$, $m \neq \max(S)$ so thus, we have a contradiction and the number of rationals between a and b must be infinite. □

Problem 8. *Ross 4.14*

Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$.
(a) Prove sup$(A + B) =$ sup$(A) +$ sup(B).
(b) Prove inf$(A + B) =$ inf$(A) +$ inf(B).

Proof.

$$\begin{aligned} a &\leq \sup(A)\ \forall a \in A \\ a + b &\leq \sup(A) + b\ \forall a \in A \forall b \in B \\ a + b &\leq \sup(A) + \sup(B)\ \forall a \in A \forall b \in B \\ \sup(A + B) &\leq \sup(A) + \sup(B). \end{aligned}$$

$$\begin{aligned} a + b &\leq \sup(A + B)\ \forall a \in A \forall b \in B \\ a &\leq \sup(A + B) - b\ \forall a \in A \forall b \in B \\ \sup(A) &\leq \sup(A + B) - b\ \forall b \in B \\ b &\leq \sup(A + B) - \sup(A)\ \forall b \in B \\ \sup(B) &\leq \sup(A + B) - \sup(A) \\ \sup(A) + \sup(B) &\leq \sup(A + B). \end{aligned}$$

$$\sup(A + B) = \sup(A) + \sup(B).$$

By similar logic, we have

$$\inf(A + B) = \inf(A) + \inf(B).$$

□

Problem 9. *Ross 7.5*

Determine the following limits.
(a) lim$(\sqrt{n^2 + 1} - n)$.
(b) lim$(\sqrt{n^2 + n} - n)$.
(c) lim$(\sqrt{4n^2 + n} - 2n)$.

$$\lim_{n\to\infty} \sqrt{n^2+1} - n = \lim_{n\to\infty} (\sqrt{n^2+1} - n)\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n}$$
$$\to 0.$$
$$\lim_{n\to\infty} \sqrt{n^2+n} - n = \lim_{n\to\infty} (\sqrt{n^2+n} - n)\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n}$$
$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{2}.$$
$$\lim_{n\to\infty} \sqrt{4n^2+n} - 2n = \lim_{n\to\infty} (\sqrt{4n^2+n} - 2n)\frac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n}$$
$$= \lim_{n\to\infty} \frac{n}{\sqrt{4n^2+n}+2n} = \lim_{n\to\infty} \frac{1}{\sqrt{4+\frac{1}{n}}+2} \to \frac{1}{4}.$$