MATH 104 HW #2

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Problem 1. *Ross 9.9*

Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.
(a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.
(b) Prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$.
(c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$.

Proof. If $\lim s_n = +\infty$ then for $\forall M > 0, \exists N_s > 0$ such that for $\forall n > N_s, s_n > M$. If for M we let $N_t = max(N_s, N_0)$, then $\forall n > N_t, t_n \geq s_n > M$ implies that $\lim t_n = +\infty$.

Likewise, if $\lim t_n = -\infty$ then for $\forall M < 0, \exists N_t > 0$ such that for $\forall n > N_t, t_n < M$. If for M we let $N_s = max(N_t, N_0)$, then $\forall n > N_s, s_n \leq t_n < M$ implies that $\lim s_n = -\infty$.

Finally, let us consider the limit of the sequence $\lim t_n - s_n = L$. Because we know that $\lim s_n$ and $\lim t_n$ exist, L also must exist. Suppose that $L < 0$. Then, $\forall \varepsilon > 0 \exists N > 0$ such that $\forall n > N, |t_n - s_n - L| < \varepsilon$. Because this must hold $\forall \varepsilon > 0$, it must hold for $\varepsilon = -L$. If we take $N = max(N, N_0)$, then this implies $L < t_n - s_n - L < -L \Rightarrow t_n - s_n < 0 \Rightarrow t_n < s_n$. This is a contradiction, so $L \geq 0$. Because $L = \lim t_n - \lim s_n$, $\lim t_n \geq \lim s_n$. □

Theorem 1. *Ratio Test*

For a sequence (s_n), suppose $s_n \neq 0$ and the limit $L = \lim |\frac{s_{n+1}}{s_n}|$ exists. If $L < 1$, then $\lim s_n = 0$. If $L > 1$, then $\lim s_n = +\infty$.

Proof. First, observe that $L > 0$ in a vein similar to the proof of Problem 1.c. Because L exists, we have $\forall \varepsilon > 0 \exists N > 0$ such that $\forall n > N, -\varepsilon < |\frac{s_{n+1}}{s_n}| - L < \varepsilon$. In the case that $L < 1$, consider $\varepsilon = \frac{1-L}{2}$ such that $|\frac{s_{n+1}}{s_n}| < \frac{L+1}{2}$. Let $a = \frac{L+1}{2}$ such that $L < a < 1$. We have that $\forall n > N$, $|s_{n+1}| < a|s_n|$. A trivial proof by induction on n gives us $|s_n| < a^{n-N}|s_N|$. Let us denote $b = N$. Now we have sufficient information to show that (s_n) converges to 0.

For a given $\varepsilon > 0$, let $c = b + \log_a(\varepsilon/|s_b|)$ such that $\varepsilon = a^{c-b}|s_b|$. We can now define $N = max(b, c)$ such that $\forall n > N$ we have $|s_n| < a^{n-b}|s_b| < a^{c-b}|s_b| < \varepsilon$. This implies that $\lim s_n = 0$.

In the case that $L > 1$, consider the sequence (t_n) where $t_n = \frac{1}{|s_n|}$. Then $\lim |\frac{t_{n+1}}{t_n}| = 1/L < 1$. By the above case, we have $\lim t_n = 0$. Thus, by Theorem 9.10, $\lim s_n = +\infty$. □

Problem 2. *Ross 9.15*

Show $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

Proof. Let $s_n = \frac{a^n}{n!}$. Then, we have

$$|\frac{s_{n+1}}{s_n}| = |\frac{a^{n+1}/(n+1)!}{a^n/n!}| = \frac{|a|}{n+1}.$$

By Theorem 9.2 and 9.7, $\lim |\frac{s_{n+1}}{s_n}| = 0 < 1$ so by the Ratio Test, $\lim s_n = 0$. $\square$

Problem 3. *Ross 10.7*

Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S$ is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$.

Proof. By definition of supremum, we have $\forall s \in S$, $\sup S > S$ and $\forall \varepsilon > 0 \exists s \in S, \sup S - \varepsilon < s$. Because the second condition must hold $\forall \varepsilon > 0$, consider a sequence (s_n) defined such that $s_n > \sup S - 1/n$. We can show that this sequence (s_n) converges to $\sup S$.

For a given $\varepsilon > 0$, we let $N = 1/\varepsilon$ such that $\forall n > N$, we have $s_n - \sup S > -1/n > -1/N = -\varepsilon$. Because $s_n \in S$, we also have $\sup S > s_n \implies s_n < \sup S + \varepsilon$. Thus, by definition, we have constructed a sequence (s_n) such that $\lim s_n = \sup S$. $\square$

Problem 4. *Ross 10.8*

Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove (σ_n) is an increasing sequence.

Proof. Because (s_n) is increasing, we know that $s_a > s_b$ for any $a > b$. We first show that $s_n > \sigma_{n-1}$.

$$\sigma_{n-1} = \frac{1}{n-1}\sum_{i=1}^{n-1} s_i < \frac{1}{n-1}\sum_{i=1}^{n-1} s_{n-1} = s_{n-1} < s_n.$$

We then have

$$\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i > \frac{1}{n}(\sum_{i=1}^{n-1} s_i + \sigma_{n-1}) = \frac{1}{n}(1 + \frac{1}{n-1})\sum_{i=1}^{n-1} s_i = \sigma_{n-1}.$$

Thus, (σ_n) is an increasing sequence. $\square$

Problem 5. *Ross 10.9*

Let $s_1 = 1$ and $s_{n+1} = (\frac{n}{n+1})s_n^2$ for $n \geq 1$.
(a) Find s_2, s_3 and s_4.
(b) Show $\lim s_n$ exists.
(c) Prove $\lim s_n = 0$.

- $s_2 = (\frac{1}{2})s_1^2 = \frac{1}{2}$
- $s_3 = (\frac{2}{3})s_2^2 = \frac{1}{6}$
- $s_4 = (\frac{3}{4})s_3^2 = \frac{1}{48}$

Proof. First, we demonstrate that (s_n) is a positive decreasing sequence. We can prove this using simultaneous strong induction. Clearly, the base case $s_2 < s_1$ and $s_1 > 0$ holds. Now suppose that for $1 \leq k < n$, we have $s_{k+1} < s_k$ and $s_{k+1} > 0$, which implies $0 < s_n < s_{n-1} < \cdots < s_1 = 1$. Then, $s_{n+1} = (\frac{n}{n+1})s_n^2 = (\frac{n}{n+1})(s_n)s_n < s_n$ and $s_{n+1} = (+)(+)^2 = (+)$. Thus, by induction, (s_n) is a positive decreasing sequence.

Because $s_1 = 1$, this means that (s_n) is bounded from below by 0 and bounded from above by 1. Because (s_n) is bounded and decreasing, it must be convergent. Because the limit exists, this allows us to compute the limit $\lim s_{n+1} = \lim(\frac{n}{n+1})s_n^2 \Rightarrow \lim s_n = \lim \frac{n}{n+1} \cdot \lim s_n \cdot \lim s_n = 0$. □

Problem 6. *Ross 10.10*

Let $s_1 = 1$ *and* $s_{n+1} = \frac{1}{3}(s_n + 1)$ *for* $n \geq 1$.
(a) Find s_2, s_3 *and* s_4.
(b) Use induction to show $s_n > \frac{1}{2}$ *for all* n.
(c) Show (s_n) *is a decreasing sequence.*
(d) Show $\lim s_n$ *exists and find* $\lim s_n$.

- $s_2 = \frac{1}{3}(s_1 + 1) = \frac{2}{3}$
- $s_3 = \frac{1}{3}(s_2 + 1) = \frac{5}{9}$
- $s_4 = \frac{1}{3}(s_3 + 1) = \frac{14}{27}$

Proof. First, we show that $s_n > \frac{1}{2}$ for all n. Clearly, the base case $s_1 = 1 > \frac{1}{2}$ holds. Now suppose that $s_n > \frac{1}{2}$. Then, $s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{2}$. Thus, by induction, $s_n > \frac{1}{2}$ for all n.

Now, suppose that (s_n) is not a decreasing sequence. This implies that $\exists n$ such that $s_{n+1} \geq s_n$. We then have $\frac{1}{3}(s_n + 1) \geq s_n \Rightarrow \frac{2}{3}s_n \leq \frac{1}{3} \Rightarrow s_n \leq \frac{1}{2}$. By contradiction, (s_n) must be a decreasing sequence.

Because $s_1 = 1$, this means that (s_n) is bounded from below by $\frac{1}{2}$ and bounded from above by 1. Because (s_n) is bounded and decreasing, it must be convergent. Because the limit exists, this allows us to compute the limit $\lim s_{n+1} = \lim \frac{1}{3}(s_n + 1) \Rightarrow \lim s_n = \frac{1}{3} \lim s_n + \frac{1}{3} \Rightarrow \lim s_n = \frac{1}{2}$. □

Problem 7. *Ross 10.11*

Let $t_1 = 1$ *and* $t_{n+1} = [1 - \frac{1}{4n^2}]\dot{t}_n$ *for* $n \geq 1$.
(a) Show $\lim t_n$ *exists.*
(b) What do you think $\lim t_n$ *is?*

Proof. The proof that $\lim t_n$ exists is similar to Problem 5. We first show that (t_n) is a positive decreasing sequence using simultaneous strong induction. Clearly, the base case $t_2 < t_1$ and $t_1 > 0$ holds. Now suppose that for $1 \leq k < n$, we have $t_{k+1} < t_k$ and $t_{k+1} > 0$, which implies $0 < t_n < t_{n-1} < \cdots < t_1 = 1$. Then, $t_{n+1} = (1 - \frac{1}{4n^2})t_n < 1$ and $t_{n+1} = (+)(+) = (+)$. Thus, by induction, (t_n) is a positive decreasing sequence.

Because $t_n = 1$, this means that (t_n) is bounded from below by 0 and bounded from above by 1. Because (t_n) is bounded and decreasing, it must be convergent. The limit itself is equal to an infinite product

$$\prod_{n=1}^{\infty}(1-\frac{1}{4n^2})=\prod_{n=1}^{\infty}(1-\frac{(\pi/2)^2}{n^2\pi^2})=\frac{\sin(\pi/2)}{\pi/2}=\frac{2}{\pi}.$$

□

Problem 8. *Squeeze Theorem*

Let (a_n), (b_n), (c_n) *be three sequences such that* $a_n \le b_n \le c_n$ *and* $L = \lim a_n = \lim c_n$. *Show that* $\lim b_n = L$.

Proof. By definition, $\forall \varepsilon > 0 \exists N_a, N_c > 0$ such that $\forall n > N_a, |a_n - L| < \varepsilon$ and $\forall n > N_c, |c_n - L| < \varepsilon$. Let us take $N_b = max(N_a, N_c)$ such that both statements hold true simultaneously $\forall n > N_b$. This implies $-\varepsilon < a_n - L < \varepsilon \Rightarrow -\varepsilon < b_n - L$ and $-\varepsilon < c_n - L < \varepsilon \Rightarrow b_n - L < \varepsilon$. Combinding, we get that $|b_n - L| < \varepsilon$ which implies that $\lim b_n = L$. □