MATH 104 HW #3

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Problem 1. *Ross 10.6.*
(a) *Let (s_n) be a sequence such that*

$$|s_{n+1} - s_n| < 2^{-n} \text{ for all } n \in \mathbb{N}.$$

Prove (s_n) is a Cauchy sequence and hence a convergent sequence.
(b) *Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?*

We first prove a lemma which will assist us in proving both (a) and (b).

Lemma 1.1. *Let (s_n) be a sequence and (a_n) be a strictly positive sequence such that $|s_{n+1} - s_n| < a_n$. Define a sequence (A_n) such that $A_n = \sum_{i=0}^{n} a_n$. If (A_n) is convergent then (s_n) is Cauchy.*

Proof. We claim that $|s_{n+k} - s_n| < A_{n+k-1} - A_{n-1}$. We can prove this using induction on k. The base case trivially holds from our assumption. Now assume that $|s_{n+k} - s_n| < A_{n+k-1} - A_{n-1}$. Then, we have

$$\begin{aligned} |s_{n+k+1} - s_n| &= |(s_{n+k+1} - s_{n+k}) + (s_{n+k} - s_n)| \\ &\leq |s_{n+k+1} - s_{n+k}| + |s_{n+k} - s_n| \\ &< a_{n+k} + A_{n+k-1} - A_{n-1} = A_{n+k} - A_{n-1}. \end{aligned}$$

Because (A_n) converges, let $A = \lim A_n$. Because (a_n) is strictly positive, we know that (A_n) is strictly increasing. This implies that (A_n) is bounded from above by A. Without loss of generality, suppose that for two $m, n \in \mathbb{N}$, we have $m > n$. By extension, we have $|s_m - s_n| < A_{m-1} - A_{n-1} < A - A_{n-1}$.

For $\forall \varepsilon > 0$, let N be the smallest integer such that $A_{N-1} \geq A - \varepsilon$. Observe that because $A - A_n$ is strictly decreasing and $\lim A - A_n = 0$, such an N always exists for positive ε. Then we have $\forall m, n > N$, $|s_m - s_n| < \varepsilon$. Thus, (s_n) is Cauchy. □

Proof. Endowed with this lemma, proving (a) and (b) becomes trivial, as in the case $a_n = 2^{-n}$, we have $\lim \sum_{i=0}^{n} 2^{-i} = 2$ which implies that (s_n) is a Cauchy sequence, and hence, is also convergent. However, in the case $a_n = \frac{1}{n}$, A_n diverges which implies that (s_n) is not necessarily Cauchy. □

Problem 2. *Ross 11.2. Consider the sequences defined as follows:*

$$a_n = (-1)^n, \; b_n = \frac{1}{n}, \; c_n = n^2, \; d_n = \frac{6n+4}{7n-3}.$$

(a) *For each sequence, give an example of a monotone subsequence.*
(b) *For each sequence, give its set of subsequential limits.*
(c) *For each sequence, give its* $\limsup$ *and* $\liminf$.
(d) *Which of the sequences converges? diverges to* $+\infty$*? diverges to* $-\infty$*?*
(e) *Which of the sequences is bounded?*

Proof. Consider the subsequence $a_{k_n} = a_{2n} = (-1)^{2n} = 1$. Because a_{k_n} is constant, it is monotone, and $\lim a_{k_n} = 1$. Also consider the subsequence $a_{k_n} = a_{2n+1} = (-1) \cdot (-1)^{2n} = -1$. Because a_{k_n} is constant, $\lim a_{k_n} = -1$.

We claim the set of subsequential limits S of a_n is $\{-1, 1\}$. Suppose there exists another subsequential limit a. It is easy to see that $-1 \leq a_n \leq 1$. Because a_n is bounded, it is impossible for $a = \pm\infty$. We can then conclude that there exists a increasing integer sequence (k_n) such that $\forall \varepsilon > 0 \exists N > 0 \forall n > N, |a_{k_n} - a| < \varepsilon$. Becuase $\{a_n\} = \{-1, 1\}$, we can split a_{k_n} into 3 cases.

If a_{k_n} strictly consists of 1, then $a = 1$; if a_{k_n} strictly consists of -1, then $a = -1$. If a_{k_n} consists of ± 1, then $a + \varepsilon > 1$ and $a - \varepsilon > -1$. This implies that $\varepsilon > 1$, which implies that convergence fails for $\varepsilon \leq 1$. Thus, such an a cannot exist, so therefore, $S = \{-1, 1\}$.

We can then conclude $\limsup a_n = \sup S = 1$, $\liminf a_n = \inf S = -1$. Because $\limsup a_n \neq \liminf a_n$, $\lim a_n$ is divergent. □

Proof. Because $b_{n+1} = \frac{1}{n+1} < b_n = \frac{1}{n}$, b_n is monotone decreasing. Hence, any subsequence of b_n is also monotone decreasing. It is easy to see that $\lim b_n = 0$. Thus, its set of subsequential limits is simply $S = \{0\}$ and $\limsup b_n = \liminf b_n = 0$. Because b_n is monotone decreasing and converges to 0, it is bounded from below by 0 and bounded from above by $b_1 = 1$. □

Proof. Because $c_{n+1} = (n+1)^2 > c_n = n^2$, c_n is monotone increasing. Hence, any subsequence of c_n is also monotone increasing. It is easy to see that $\lim c_n = +\infty$. Thus, its set of subsequential limits is simply $S = \{+\infty\}$ and $\limsup c_n = \liminf c_n = +\infty$. Because c_n diverges to $+\infty$, it is not bounded. □

Proof. We first show that d_n is monotone decreasing.

$$\begin{aligned} d_{n+1} \leq d_n &\Leftrightarrow \frac{6(n+1)+4}{7(n+1)-3} = \frac{6n+10}{7n+4} \leq \frac{6n+4}{7n-3} \\ &\Leftrightarrow (6n+10)(7n-3) \leq (6n+4)(7n+4) && (n \geq 1) \\ &\Leftrightarrow 42n^2 + 52n - 30 \leq 42n^2 + 52n + 16 \\ &\Leftrightarrow -30 \leq 16 && \Leftrightarrow n \geq 1. \end{aligned}$$

Because of this, any subsequence of cd_n is also monotone decreasing. We now evaluate $\lim d_n$.

$$\lim d_n = \lim \frac{6n+4}{7n-3} = \lim \frac{6 + 4/n}{7 - 3/n} = \frac{\lim 6 + 4/n}{\lim 7 - 3/n} = \frac{6}{7}.$$

Because d_n converges, its set of subsequential limits is simply $S = \{\frac{6}{7}\}$ and $\limsup d_n = \liminf d_n = \frac{6}{7}$. Because d_n is monotone decreasing and converges to $\frac{6}{7}$, it is bounded from below by $\frac{6}{7}$ and bounded from above by $d_1 = \frac{5}{2}$. $\square$

Problem 3. *Ross 11.3. Repeat Problem 2 for the sequences:*

$$s_n = \cos(\frac{n\pi}{3}),\ t_n = \frac{3}{4n+1},\ u_n = (-\frac{1}{2})^n,\ v_n = (-1)^n + \frac{1}{n}.$$

Proof. Observe that we can utilize the periodicity of cos to rewrite s_n as

$$s_n = \begin{cases} 1 & n = 6k \\ 1/2 & n = 6k+1 \text{ or } n = 6k+5 \\ -1/2 & n = 6k+2 \text{ or } n = 6k+4 \\ -1 & n = 6k+3 \end{cases},$$

where k is an integer. Thus, a simple example of a monotone subsequence is $s_{k_n} = s_{6n} = 0$. We claim the set of subsequential limits $S = \{-1, -1/2, 1/2, 1\}$. The proof of this claim is a simple proof of variations of a_n of Problem 2 via the additive and multiplicative properties of convergent sequences. We quickly find that $\limsup s_n = \sup S = 1$, $\liminf s_n = \inf S = -1$. Because $\limsup s_n \neq \liminf s_n$, $\lim s_n$ is divergent. However, $-1 \leq s_n \leq 1$ which implies s_n is bounded. $\square$

Proof. Because $t_{n+1} = \frac{3}{4(n+1)+1} = \frac{3}{4n+5} < t_n = \frac{3}{4n+5}$, t_n is monotone decreasing. Hence, any subsequence of t_n is also monotone decreasing. Evaluating $\lim t_n$, we find

$$\lim t_n = \lim \frac{3}{4n+1} = \lim \frac{3/n}{4+1/n} = \frac{\lim 3/n}{4+1/n} = 0.$$

Thus, its set of subsequential limits is simply $S = \{0\}$ and $\limsup t_n = \liminf t_n = 0$. Because t_n is monotone decreasing and converges to 0, it is bounded from below by 0 and bounded from above by $t_0 = 3$. $\square$

Proof. Consider the subsequence $u_{k_n} = u_{2n} = (\frac{1}{4})^n = 4^{-n}$. Clearly, $4^{-(n+1)} = \frac{4^{-n}}{4} < 4^{-n}$, so this subsequence is monotone decreasing. We demonstrate that $\lim u_n = 0$. First, observe that $|u_n|$ is monotone decreasing since $|u_{n+1}| = (\frac{1}{2})^{n+1} < |u_n| = (\frac{1}{2})^n$. Thus, for any $\varepsilon > 0$, let $N = max(0, \log_{1/2}(\varepsilon))$. Then we have for any $n > N$, $|u_n| < |u_N| < \varepsilon$. Thus, u_n converges to 0, its set of subsequential limits is simply $S = \{0\}$, and $\limsup u_n = \liminf u_n = 0$. Because $|u_n|$ is monotone decreasing, it is bounded from above by the first positive term $u_0 = 1$ and bounded from below by the first negative term $u_1 = -1/2$. $\square$

Proof. Consider the subsequence $v_{k_n} = v_{2n} = 1 + \frac{1}{n}$. Clearly, $1 + \frac{1}{n+1} < 1 + \frac{1}{n}$, so this subsequence is monotone decreasing.

We claim the set of subsequential limits $S = \{-1, 1\}$. Suppose there exists another subsequential limit v. We first check that v_n is bounded by observing that it is the sum of two bounded sequences. Because v_n is bounded, it is impossible for $v = \pm\infty$.

We can then conclude that there exists a increasing integer sequence (k_n) such that $\forall \varepsilon > 0 \exists N > 0 \forall n > N, |v_{k_n} - v| < \varepsilon$. This implies that $v - \varepsilon - \frac{1}{k_n} < (-1)^{k_n} < v + \varepsilon - \frac{1}{k_n}$. Because $(-1)^{k_n}$ can only take values in $\{-1, 1\}$, we have $v - \varepsilon - \frac{1}{k_n} < -1, v + \varepsilon - \frac{1}{k_n} > 1$. This implies that $\varepsilon > 1$, which implies that convergence fails for $\varepsilon \leq 1$. Thus, such an v cannot exist, so therefore, $S = \{-1, 1\}$.

We can then conclude $\limsup v_n = \sup S = 1$, $\liminf v_n = \inf S = -1$. Because $\limsup v_n \neq \liminf v_n$, $\lim v_n$ is divergent. □

Problem 4. *Ross 11.5. Let* (q_n) *be an enumeration of all the rationals in the interval* $(0, 1]$.
(a) *Give the set of subsequential limits for* (q_n).
(b) *Give the values of* $\limsup q_n$ *and* $\liminf q_n$.

Proof. We claim that the set of subsequential limits for (q_n) is $[0, 1]$. We first prove that if $q \in [0, 1]$, then $q \in S$. In order to prove this, we define an inductive algorithm which defines a subsequence $a_n = q_{k_n}$ which converges to q.

For our base case, we let p_0 be any arbitrary rational in $\{q_n\}$ with finite index i_p such that $p_0 = q_{i_p} \neq q$ and $k_0 = i_p$.

Now suppose that we have defined k_n up to k_j. Let $p = q_{k_j}$ and $i_p = k_j$. Here, it may seem natural to utilize the denseness of $\mathbb{Q}$ to construct a new member of the subsequence. However, this condition alone is not sufficient to create a subsequence which converges. Instead, we let $r = (p + q)/2$ with finite index i_r such that if $q < p$, we have $q < r < p$ and if $p < q$, we have $p < r < q$. It is easy to show that r is rational as

$$r = \frac{p + q}{2} = \frac{c_1/d_1 + c_2/d_2}{2} = \frac{c_1 d_2 + c_2 d_1}{2 d_1 d_2}.$$

If $i_r > i_p$, then $k_{j+1} = i_r$, and the inductive step advances forward to k_{j+1}. If $i_r < i_p$, then we set $r' = (r + q)/2$ with finite index i'_r. We then repeat the current inductive step for r' instead of r. Because there are only a finite number of indicies smaller than i_p, we must eventually find an r' such that $i'_r > i_p$, which completes the inductive algorithm.

If $p_0 < q$, our algorithm gives us a monotone increasing sequence with $q > a_n > p_0$, and if $p_0 > q$, our algorithm gives us a monotone decreasing sequence with $q < a_n < p_0$. This implies that in both cases, our sequences are convergent. An essential observation to make is that because our inductive step may repeat, the next term in the sequence may be a successive average of the previous term with q. This implies that if $p_0 < q$, we have the additional condition that $a_{n+1} > \frac{a_n + q}{2}$, and if $p_0 > q$, we have the additional condition that $a_{n+1} < \frac{a_n + q}{2}$.

In the first case, our additional condition implies $a_{n+1} > q - (a_{n+1} - a_n)$. For any $\varepsilon > 0$, let $a_n - a_{n-1} < \varepsilon$. Because (a_n) is monotone increasing and convergent, we know that (a_n) is also Cauchy. This implies that it is always possible to find such an n which satisfies the ε condition. This implies $\forall \varepsilon >$

$0 \exists n > 0$ such that $a_n > q - \varepsilon$, which implies that q is a least upper bound. This implies that $\lim q_{k_n} = \lim a_n = q$.

The proof for the second case follows similarly, utilizing the additional condition and Cauchy condition to show that q is the greatest lower bound which implies $\lim q_{k_n} = \lim a_n = q$.

Now, we show that if $q \notin [0, 1]$, then $q \notin S$. Suppose that $q \in S$. This implies that there exists a positive, increasing sequence of integers (k_n) such that $\forall \varepsilon > 0 \exists N > 0 \forall n > N, |q_{k_n} - q| < \varepsilon$. We can split the proof into two cases: 1) $q < 0$ and 2) $q > 1$.

If $q < 0$, then using the fact that $q_{k_n} - q < \varepsilon$ and $q_{k_n} > 0$ we have $-q < \varepsilon$ which implies that our condition fails for $\varepsilon < -q$, which is a contradiction. If $q > 1$, then using the fact that $-\varepsilon < q_{k_n} - q$ and $q_{k_n} \leq 1$ we have $-\varepsilon < 1 - q$ which implies that our condition fails for $\varepsilon < q - 1$, which is a contradiction. Thus, $q \notin S$.

Because $S = [0, 1]$, we have $\limsup q_n = 1$ and $\liminf q_n = 0$. □

Discussion 1. *What exactly is* $\limsup$*?*

For some sequence (a_n), $\limsup a_n$ is formally defined as $\lim_{N \to \infty} \sup\{a_n : n > N\}$. However, the meaning of $\limsup$ is lost in the formal definition. Instead, it is easier to think of $\limsup$ and $\liminf$ as the upper and lower bounds of a sequence for sufficiently large n. In other words, it characterizes the long-run behavior of a sequence by giving us a range of possible values of a_n.

One important note is that $\limsup a_n \neq \sup\{a_n\}$ and $\liminf a_n \neq \inf\{a_n\}$. Consider the sequence $(a_n) = 1, -1, 0, 0, ...$ Clearly, $\limsup a_n = \liminf a_n = 0$ because the only possible value of a_n for sufficiently large n (in this case, simply $n \geq 2$) is 0. However, $\sup\{a_n\} = 1$ and $\inf\{a_n\} = -1$.

Instead, it is more appropriate to think of $\limsup a_n$ as $\inf \sup a_n$, or the smallest supremum of the successive elements of the sequence, and $\liminf a_n$ as $\sup \inf a_n$, or the largest infimum of the sucessive elements of the sequence. Formally, this is written as $\limsup a_n = \inf\{\sup\{a_n : n > N\} : N \geq 0\}$ and $\liminf a_n = \sup\{\inf\{a_n : n > N\} : N \geq 0\}$.