MATH 104 HW #4

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Question 1. *In Cantor's diagonalization argument, we construct a subsequence by selecting elements from a collection of subsequences, using the fact that there are an infinite number of elements in (a_n) in the neighborhood of some $s \in R$. How do we know that the indicies $n_{11} < n_{22} < \cdots$?*

Question 2. *When evaluating series, it is usually proper to include the $n = 0$ (constant) term; however, in sequences, this is usually not the case. In general, should series and sequences be indexed starting at $n = 0$ or $n = 1$? (Of course, this is excluding the cases where a sequence or series is not defined at some n.)*

Question 3. *How is it valid that we can use the integral test to prove the convergence or divergence of a series if we have not clearly defined the concept of continuity or functions?*

Question 4. *How do we define convergence for complex power series?*

Question 5. *Suppose we have a sequence which is bounded by two monotone sequences. Does this sequence satisfy all the properties of monotone sequences (except monotonicity)?*

Problem 1. *Ross 12.10. Prove (s_n) is bounded if and only if $\limsup |s_n| < +\infty$.*

Proof. We first prove the forward direction. Suppose (s_n) is bounded but $\limsup |s_n| = +\infty$. This implies that $\alpha \leq s_n \leq \beta$. The latter implies that there exists a subsequence (s_{n_k}) of (s_n) such that $\lim_{k\to\infty} |s_{n_k}| = +\infty$. This implies that $\forall M > 0 \exists N > 0$ such that $\forall k > N, |s_{n_k}| > M$. If we let $M = max(|\alpha|, |\beta|)$, then there exists k such that $|s_{n_k}| > \beta$ or $s_{n_k} < \alpha$. This is a contradiction, so $\limsup |s_n| < +\infty$.

Next we prove the reverse direction. Because $\limsup |s_n| < +\infty$, let $L = \limsup |s_n|$. This implies that $\forall \varepsilon > 0 \exists N > 0$ such that $L + \varepsilon > \sup\{|s_n| : n > N\}$. Choosing an arbitrary ε, we find an N such that $\forall n > N, |s_n| < L + \varepsilon \Rightarrow -L - \varepsilon < s_n < L + \varepsilon$. Because the elements $\{|s_n| : n \leq N\}$ form a finite subset of $\mathbb{R}$, we know that this set must be bounded. This implies (s_n) is bounded. □

Problem 2. *Ross 12.12. Let (s_n) be a sequence of nonnegative numbers, and for each n define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.*
(a) Show

$$\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n.$$

(b) Show that if $\lim s_n$ exists then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.
(c) Give an example where $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.

Proof. We begin by showing that for $M > N$,

$$\sup\{\sigma_n : n \geq M\} \leq \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}.$$

We know that $\sup\{s_n : n > N\} \geq s_n \forall n > N$. Consider the set $S = \{s_n : N < n \leq M\}$. Because S is a finite subset of $\mathbb{R}$, it has a maximum, which we shall denote $s_k \in S$. We then get

$$\begin{aligned}
\sup\{s_n : n > N\} + \frac{1}{M}(s_1 + s_2 + \cdots + s_N) &\geq \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + s_k \\
&\geq \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \frac{M-N}{M}s_k + \frac{N}{M}s_k \\
&\geq \frac{1}{M}(s_1 + s_2 + \cdots + S_N) + \frac{1}{M}(s_{N+1} + \cdots + s_M) + \frac{N}{M}s_k \\
&= \sigma_M + \frac{N}{M}s_k \geq \sup\{\sigma_n : n \geq M.\}
\end{aligned}$$

Taking the limit $N \to \infty$, this implies $\limsup \sigma_n \leq \limsup s_n$. Similarly, using inf and reversing the direction of the inequality, we find $\liminf s_n \leq \liminf \sigma_n$. Because $\liminf \sigma_n \leq \limsup \sigma_n$ by definition, this gives us

$$\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n.$$

If $\lim s_n$ exists, let $\lim s_n = \limsup s_n = \liminf s_n = L$. Then, we have $L \leq \liminf \sigma_n \leq L$ and $L \leq \limsup \sigma_n \leq L$ which implies $\lim \sigma_n = \limsup \sigma_n = \liminf \sigma_n = L$.

Notice that the reverse direction does not hold. Namely, if we let $s_n = 1$ if n is odd and $s_n = -1$ if n is even, then $\sigma_n = \frac{1}{n}$ if n is odd and $\sigma_n = 0$ if n is even. Although $\lim \sigma_n = 0$, $\lim s_n$ does not exist. □

Problem 3. *Ross 14.2. Determine which of the following series converge. Justify your answers.*
(a) $\sum \frac{n-1}{n}$, *(b)* $\sum (-1)^n$, *(c)* $\sum \frac{3n}{n^3}$, *(d)* $\sum \frac{n^3}{3^n}$, *(e)* $\sum \frac{n^2}{n!}$, *(f)* $\sum \frac{1}{n^n}$, *(g)* $\sum \frac{n}{2^n}$.

- $\lim \frac{n-1}{n} = 1 \Rightarrow \sum \frac{n-1}{n}$ diverges.
- $\lim (-1)^n$ diverges $\Rightarrow \sum (-1)^n$ diverges.
- $\sum \frac{3n}{n^3} = 3 \sum \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges, $\sum \frac{3n}{n^3}$ also converges.
- By the Root Test, $\lim(|\frac{n^3}{3^n}|)^{1/n} = \frac{1}{3}\lim(n^{1/n})^3 = \frac{1}{3} < 1$. Thus, $\sum \frac{n^3}{3^n}$ converges.
- By the Ratio Test, $\lim \frac{|(n+1)^2/(n+1)!|}{|n^2/n!|} = \lim(\frac{n+1}{n})^2 \frac{1}{n+1} = 0$. Thus, $\sum \frac{n^2}{n!}$ converges.
- By the Root Test, $\lim(|\frac{1}{n^n}|)^{1/n} = \lim \frac{1}{n} = 0$. Thus, $\sum \frac{1}{n^n}$ converges.
- By the Ratio Test, $\lim \frac{|(n+1)/2^{n+1}|}{|n/2^n|} = \frac{1}{2}\lim \frac{n+1}{n} = \frac{1}{2}$. Thus, $\sum \frac{n}{2^n}$ converges.

Problem 4. *Ross 14.10. Find a series $\sum a_n$ which diverges by the Root Test but for which the Ratio Test gives no information.*

Proof. Consider the sequence $a_n = 1$ if n is odd and $a_n = 2^n$ if n is even. Then, $\liminf \frac{|a_{n+1}|}{|a_n|} = \lim \frac{1}{1} = 1$ and $\limsup \frac{|a_{n+1}|}{|a_n|} \lim \frac{2^{n+1}}{2^n} = 2$. Because $\liminf \frac{|a_{n+1}|}{|a_n|} \leq 1 \leq \limsup \frac{|a_{n+1}|}{|a_n|}$, the Ratio Test is inconclusive. However, $\limsup(|a_n|)^{1/n} = \lim(2^n)^{1/n} = 2$, which implies $\sum a_n$ diverges. □

Problem 5. *Rudin 3.6. Investigate the behavior of $\sum a_n$ if*
(a) $a_n = \sqrt{n+1} - \sqrt{n}$;
(b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$;
(c) $a_n = (\sqrt[n]{n} - 1)^n$;
(d) $a_n = \frac{1}{1+z^n}$, for complex values of z.

Proof. Consider the sequence of partial sums $s_n = \sum_{i=0}^n a_i$. We get

$$\begin{aligned} s_n &= \sum_{i=0}^{n} \sqrt{i+1} - \sqrt{i} = \sum_{i=0}^{n} \sqrt{i+1} - \sum_{i=0}^{n} \sqrt{i} = \sum_{i=1}^{n+1} \sqrt{i} - \sum_{i=0}^{n} \sqrt{i} \\ &= \sqrt{n+1}. \end{aligned}$$

This implies $\sum a_n = \lim s_n = +\infty \Rightarrow \sum a_n$ diverges. □

Proof. Consider the sequence of partial sums $s_n = \sum_{i=0}^n a_i$. We get

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{\sqrt{i+1} - \sqrt{i}}{i} = \sum_{i=1}^{n} \frac{\sqrt{i+1}}{i} - \sum_{i=1}^{n} \frac{\sqrt{i}}{i} = \sum_{i=1}^{n+1} \frac{\sqrt{i}}{i-1} - \sum_{i=1}^{n} \frac{\sqrt{i}}{i} \\ &= \frac{\sqrt{n+1}}{n} - 1 + \sum_{i=2}^{n} \sqrt{i}\left(\frac{1}{i-1} - \frac{1}{i}\right) = \frac{\sqrt{n+1}}{n} - 1 + \sum_{i=2}^{n} \frac{\sqrt{i}}{i(i-1)}. \end{aligned}$$

Because

$$\lim \frac{\sqrt{n+1}}{n} = \lim \frac{\sqrt{n}}{n+1} = \lim \frac{1}{\sqrt{n} + 1/\sqrt{n}} = 0,$$

it suffices to investigate the behavior of $\sum \frac{\sqrt{n}}{n(n-1)}$. Rewriting, we get

$$\frac{\sqrt{n}}{n(n-1)} = \frac{1}{n^{3/2}} \cdot \frac{1}{1 - 1/n} \leq \frac{2}{n^{3/2}}.$$

Because $\sum \frac{1}{n^{3/2}}$ converges, this implies $\sum a_n$ converges. □

Proof. By the Root Test, $\lim(|(\sqrt[n]{n} - 1)^n|)^{1/n} = \lim \sqrt[n]{n} - 1 = 0$. Thus, $\sum a_n$ converges. □

Proof. Let us express $z = re^{i\theta}$, where r is positive and $0 \leq \theta \leq 2\pi$. If $r \leq 1$, then $\lim a_n = \lim \frac{1}{1+r^n e^{i\theta n}} = 1$, which implies $\sum a_n$ diverges. If $r > 1$, then because $\frac{1}{1+r^n e^{i\theta n}} \leq \frac{1}{r^n e^{i\theta n}} = r^{-n} e^{i\bar{\theta} n}$ and $\sum \frac{1}{a^n}$ converges, $\sum a_n$ also converges. □

Problem 6. *Rudin 3.7. Prove that the convergence of $\sum a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$ if $a_n \geq 0$.*

Proof. Observe that we can apply the simple inequality $(a+b)^2 \geq 2ab$ to get $(a_n + n^{-2})^2 \geq 2a_n n^{-2} \Rightarrow (a_n + n^{-2})/\sqrt{2} \geq \sqrt{a_n} n^{-1}$. Because both $\sum a_n$ and $\sum n^{-2}$ converge, $\sum \frac{\sqrt{a_n}}{n}$ also must converge. □

Problem 7. *Rudin 3.9. Find the radius of convergence of each of the following power series:*
(a) $\sum n^3 z^n$, (b) $\sum \frac{2^n}{n!} z^n$, (c) $\sum \frac{2^n}{n^2} z^n$, (d) $\sum \frac{n^3}{3^n} z^n$.

For the following problems, we only evaluate the power series based on the modulus of z.

Proof. Observe that $\lim n^{1/n} = 1 \Rightarrow \lim(n^3)^{1/n} = \lim(n^{1/n})^3 = 1^3 = 1 \Rightarrow \limsup(n^3)^{1/n} = 1$. Thus, $\alpha = \limsup |a_n|^{1/n} = \limsup(n^3)^{1/n} = 1 \Rightarrow R = 1$. If $z = \pm 1$, then the series is equivalent to $\sum(\pm 1)^n$ which diverges. Thus, our radius of convergence is $-1 < z < 1$. □

Proof. We first compute the limit $\lim \frac{1}{(n!)^{1/n}}$. We know that $\frac{1}{(n!)^{1/n}}$ is bounded from below by 0. Consider that in the product expansion of $n!$, there are at least $n/2$ terms greater than $n/2$. Thus, $n! \geq (\frac{n}{2})^{n/2}$. This implies

$$\lim \frac{1}{(n!)^{1/n}} \leq \lim \frac{1}{(n/2)^{1/2}} = \lim \frac{\sqrt{2}}{\sqrt{n}} = 0.$$

Thus,

$$\alpha = \limsup |a_n|^{1/n} = \limsup(\frac{2^n}{n!})^{1/n} = \lim \frac{2}{(n!)^{1/n}} = 0 \Rightarrow R = +\infty.$$

Therefore, our radius of convergence is all real numbers. □

Proof. Similar to the first series, $\alpha = \limsup |a_n|^{1/n} = \limsup(\frac{2^n}{n^2})^{1/n} = \limsup \frac{2}{(n^2)^{1/n}} = 2 \Rightarrow R = \frac{1}{2}$. If $z = \pm\frac{1}{2}$, then the series is equivalent to $\sum(\pm 1)^n \frac{1}{n^2}$ which converges. Thus, our radius of convergence is $-\frac{1}{2} \leq z \leq \frac{1}{2}$. □

Proof. Similar to the first series, $\alpha = \limsup |a_n|^{1/n} = \limsup(\frac{n^3}{3^n})^{1/n} = \limsup \frac{(n^3)^{1/n}}{3} = \frac{1}{3} \Rightarrow R = 3$. If $z = \pm 3$, then the series is equivalent to $\sum(\pm 1)^n n^3$ which diverges. Thus, our radius of convergence is $-3 < z < 3$. □

Problem 8. *Rudin 3.12. Suppose $a_n > 0, s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.*
(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.
(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ *diverges.*
(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ *converges.*
(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \text{ and } \sum \frac{a_n}{1+n^2a_n}?$$

Proof. Suppose that $\lim a_n$ diverges to $\pm\infty$. Then $\lim \frac{a_n}{a_n+1} = 1$, which implies $\sum \frac{a_n}{a_n+1}$ also diverges.

If $\lim a_n$ diverges otherwise, then let $\limsup a_n = a$ and $\liminf b_n = b, a \neq b$. This implies there exists subsequences of a_n such that their limits equal a and b, which further implies that there exists subsequences of $\frac{a_n}{a_n+1}$ such that their limits equal $\frac{a}{a+1}$ and $\frac{b}{b+1}$. Hence, $\sum \frac{a_n}{a_n+1}$ also diverges.

If $\lim a_n$ exists and is nonzero, then $\lim \frac{a_n}{a_n+1}$ is also nonzero, which implies $\sum \frac{a_n}{a_n+1}$ also diverges.

This leaves us with the case that $\lim a_n$ exists and equals 0. If this is the case, consider the limit

$$\lim \frac{a_n}{\frac{a_n}{a_n+1}} = \lim a_n + 1 = 1.$$

This implies that the tail end behavior of $\frac{a_n}{a_n+1}$ is equivalent to a_n, which implies $\frac{a_n}{a_n+1}$ diverges.

Now consider the series $\sum \frac{a_n}{s_n}$. We demonstrate its divergence by first proving that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}.$$

Because a_n is positive, s_n is increasing. This implies that

$$\begin{aligned}\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \\ &\geq \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}.\end{aligned}$$

Using this with the fact that s_n is strictly increasing, we can show that

$$|\sum_{k=n}^{p} \frac{a_k}{s_k}| = \sum_{k=n}^{p} \frac{a_k}{s_k} \geq 1 - \frac{s_{n-1}}{s_p} > 0$$

which implies that $\sum \frac{a_n}{s_n}$ fails the Cauchy criterion and is hence divergent.

Now consider the series $\sum \frac{a_n}{s_n^2}$. We demonstrate its convergence by first showing that

$$\frac{a_n}{s_n} \leq \frac{a_n}{s_{n-1}} = \frac{s_n}{s_{n-1}} - 1 \Rightarrow \frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}.$$

This gives us

$$|\sum_{k=n}^{p} \frac{a_k}{s_k^2}| = \sum_{k=n}^{p} \frac{a_k}{s_k^2} \leq \sum_{k=n}^{p} \frac{1}{s_{k-1}} - \frac{1}{s_k} = \sum_{k=n}^{p} \frac{1}{s_{k-1}} - \sum_{k=n}^{p} \frac{1}{s_k}$$
$$= \sum_{k=n-1}^{p-1} \frac{1}{s_k} - \sum_{k=n}^{p} \frac{1}{s_k} = \frac{1}{s_{n-1}} - \frac{1}{p} \leq \frac{1}{s_{n-1}}.$$

Now suppose that $\exists \varepsilon > 0$ such that $\forall n > 0, \varepsilon \leq \frac{1}{s_{n-1}}$. This would imply $s_{n-1} \leq \frac{1}{\varepsilon}$ is bounded from above. Because we know s_n is also bounded from below by 0, and because s_n is monotone, this implies that s_n converges. However, this would mean that $\sum a_n$ converges which is a contradiction. Thus, $\forall \varepsilon > 0 \exists n > 0$ such that $\varepsilon > \frac{1}{s_{n-1}}$. This implies that $\sum \frac{a_n}{s_n^2}$ satisfies the Cauchy criterion and hence is convergent.

Now consider the series $\sum \frac{a_n}{1+na_n}$. We can reduce $\frac{a_n}{1+na_n} = \frac{1}{n+1/a_n}$. For $n \geq 1/a_n$, $\frac{a_n}{1+na_n} \geq \frac{1}{2/a_n} = \frac{a_n}{2}$. For $n \leq 1/a_n$, $\frac{a_n}{1+na_n} \geq \frac{1}{2n}$. This implies that $\sum \frac{a_n}{1+na_n} \geq \sum \frac{a_n}{2} + \sum \frac{1}{2n}$. Since both subseries diverge, $\sum \frac{a_n}{1+na_n}$ also diverges.

Finally consider the series $\sum \frac{a_n}{1+n^2a_n}$. Because $\frac{a_n}{1+n^2a_n} \leq \frac{a_n}{n^2a_n} = \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges, $\sum \frac{a_n}{1+n^2a_n}$ also converges. $\square$