MATH 104 HW #5

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Problem 1. *Ross 13.3. Let B be the set of all bounded sequences $\boldsymbol{x} = (x_1, x_2, \cdots)$, and define $d(\boldsymbol{x}, \boldsymbol{y}) = \sup\{|x_j - y_j| : j = 1, 2, \cdots\}$.*
(a) Show d is a metric for B.
(b) Does $d^*(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{\infty} |x_j - y_j|$ define a metric for B?

Proof. We demonstrate that d satisfies the metric properties for B.

$d(\boldsymbol{x}, \boldsymbol{x}) = \sup\{|x_j - x_j| : j = 1, 2, \cdots\} = \sup\{0 : j = 1, 2, \cdots\} = 0$.

Additionally, if $d(\boldsymbol{x}, \boldsymbol{y}) = 0 \Rightarrow \sup\{|x_j - y_j| : j = 1, 2, \cdots\} = 0 \Rightarrow |x_j - y_j| \leq 0$ for $j = 1, 2, \cdots$. Because $|x_j - y_j| \geq 0$, $|x_j - y_j| = 0 \Rightarrow x_j = y_j \Rightarrow \boldsymbol{x} = \boldsymbol{y}$.

This implies that if $\boldsymbol{x} \neq \boldsymbol{y}$, then $d(\boldsymbol{x}, \boldsymbol{y}) > 0$. Because $\boldsymbol{x}$ and $\boldsymbol{y}$ are bounded, $\exists \alpha, \beta$ such that $|x_j| \leq \alpha, |y_j| \leq \beta$ for $j = 1, 2, \cdots$. Thus, $|x_j - y_j| \leq \alpha + \beta$. Because $d(\boldsymbol{x}, \boldsymbol{y}) = \sup\{|x_j - y_j| : j = 1, 2, \cdots\}$ is the least upper bound of $|x_j - y_j|$, this implies $d(\boldsymbol{x}, \boldsymbol{y})$ is finite.

Finally, by the Triangle Inequality for the metric space $(\mathbb{R}, d(x, y) = |x - y|)$,

$$\begin{aligned} d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z}) &= \sup\{|x_j - y_j| : j = 1, 2, \cdots\} + \sup\{|y_j - z_j| : j = 1, 2, \cdots\} \\ &\geq \sup\{|x_j - y_j| + |y_j - z_j| : j = 1, 2, \cdots\} \\ &\geq \sup\{|x_j - z_j| : j = 1, 2, \cdots\} = \sup d(\boldsymbol{y}, \boldsymbol{z}). \end{aligned}$$

Observe that $d^*(\boldsymbol{x}, \boldsymbol{y})$ is not a valid metric because $d^*(\boldsymbol{x}, \boldsymbol{y})$ can be infinite. To see this simple fact, consider $x_i = 1$, $y_i = 0$ such that $\boldsymbol{x}, \boldsymbol{y}$ are bounded from above and below by 1 and 0, respectively. However, $d^*(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{\infty} 1 = +\infty$. □

Problem 2. *Ross 13.5.*
(a) Verify one of DeMorgan's Laws for sets:

$$\bigcap\{S \backslash U : U \in \mathcal{U}\} = S \backslash \bigcup\{U : U \in \mathcal{U}\}.$$

(b) Show that the intersection of any collection of closed sets is a closed set.

Proof.

$$\begin{aligned} & x \in \bigcap\{S \backslash U : U \in \mathcal{U}\} \\ \Leftrightarrow\ & x \in S \backslash U \ \forall U \in \mathcal{U} \\ \Leftrightarrow\ & x \in S, x \notin U \ \forall U \in \mathcal{U} \\ \Leftrightarrow\ & x \in S, x \notin \bigcup\{U : U \in \mathcal{U}\} \\ \Leftrightarrow\ & x \in S \backslash \bigcup\{U : U \in \mathcal{U}\}. \end{aligned}$$

Thus, we get the desired result

$$\bigcap\{S\backslash U : U \in \mathcal{U}\} = S\backslash \bigcup\{U : U \in \mathcal{U}\}.$$

For some metric space (X, d), we know that $\forall V \subseteq X$ which is closed, there $\exists U \subseteq X$ which is open such that $V = X\backslash U$. Thus, applying our proven result to some collection of closed subsets $\mathcal{V}$, we have

$$\bigcap\{V : V \in \mathcal{V}\} = \bigcap\{X\backslash U : U \in \mathcal{U}\} = X\backslash \bigcup\{U : U \in \mathcal{U}\}.$$

Because the union of any collection of open subsets is open, $\bigcap\{V : V \in \mathcal{V}\}$ is closed. $\square$

Problem 3. *Ross 13.7. Show that every open set in $\mathbb{R}$ is the disjoint union of a finite or infinite sequence of open intervals.*

Proof. For $S \subseteq \mathbb{R}$ and any $x \in S$, define $a_x = \inf\{y \in \mathbb{R} : (y, x) \subseteq S\}, b_x = \sup\{y \in \mathbb{R} : (x, y) \subseteq S\}$. For now, assume that a_x, b_x are finite.

Define the interval $I_x := (a_x, b_x)$. Becuase $a_x \leq y \forall (y, x) \subseteq S$ and by definition, $y < x$, we have $a_x < x$. Similarly, because $b_x \geq y \forall (x, y) \subseteq S$ and by definition, $y > x$, we have $b_x > x$. Thus, $x \in I_x$. Because $I_x = (a_x, x) \cup \{x\} \cup (x, b_x)$ and $(a_x, x), (x, b_x) \subseteq S$, we have $I_x \subseteq S$.

Let $\mathcal{I} = \{I_x : x \in S\}$. We claim that $S = \bigcup_{I_y \in \mathcal{I}} I_y$. We first prove the forward direction. If $x \in S$, then $x \in I_x \subseteq \bigcup_{I_y \in \mathcal{I}} I_y \Rightarrow x \in \bigcup_{I_y \in \mathcal{I}} \Rightarrow S \subseteq \bigcup_{I_y \in \mathcal{I}}$. Next, we prove the reverse direction. If $x \in \bigcup_{I_y \in \mathcal{I}}$, then $\exists I_y \in \mathcal{I}, x \in I_y \subseteq S \Rightarrow x \in S \Rightarrow \bigcup_{I_y \in \mathcal{I}} \subseteq S$.

Now that we have proven that S can be written as a union of open intervals, we show that the union is disjoint. Suppose there exists $I_x, I_y \in \mathcal{I}, I_x \neq I_y$, and $z \in S$ such that $z \in I_x \cap I_y$. In other words, $\max(a_x, a_y) < z < \min(b_x, b_y)$. Now consider the interval $I_z \in \mathcal{I}$. Because $(a_x, z) \subseteq I_x \subseteq S$ and $(a_y, z) \subseteq I_y \subseteq S, a_z \leq a_x$ and $a_z \leq a_y$. Similarly, because $(z, b_x) \subseteq I_x \subseteq S$ and $(z, b_y) \subseteq I_y \subseteq S, b_z \geq b_x$ and $b_z \geq b_y$. Thus, $I_x \subseteq I_z$ and $I_y \subseteq I_z$. However, this implies $x \in I_z$ and $y \in I_z$. Using identical logic above, we find that $I_z \subseteq I_x$ and $I_z \subseteq I_y$, which implies $I_x = I_y = I_z$. This is a contradiction, so therefore, the union must be disjoint.

Returning to the assumption that a_x, b_x is finite, suppose $\exists x \in S$ such that a_x, b_x are infinite for some $x \in S$. Then, if $a_x = -\infty$, then we can express S as the disjoint union $S = S' \cup (-\infty, b_x)$, where S' is bounded from below. Likewise, if $b_x = +\infty$, then we can express S as the disjoint union $S = S' \cup (a_x, +\infty)$, where S' is bounded from above. If both $a_x = -\infty$ and $b_x = +\infty$, then $S = (-\infty, +\infty)$. $\square$

Problem 4. *Show that the closure of a closed set $\bar{S}$ is $\bar{S}$.*

Proof. We want to show that for a metric space (X, d) and $\bar{S} \in X$, $\bar{\bar{S}} = \bar{S}$, where $\bar{S} := \{x : \exists (s_n) \in S, (s_n) \to x\}$.

We first prove the forward direction. If $x \in \bar{\bar{S}}$ then $\exists (s_n) \in \bar{S}, (s_n) \to x$. Similarly, $\forall s_n \exists (t_k) \in S, (t_k) \to s_n$. From these, we get that $\forall \varepsilon_s > 0 \exists N_s > 0$ such that $\forall n > N_s, d(s_n, x) < \varepsilon_s$ and $\forall s_n \forall \varepsilon_t > 0 \exists N_t > 0$ such that $\forall k > N_t, d(t_k, s_n) < \varepsilon_t$. If we let $\varepsilon_s = \varepsilon_t = \varepsilon/2$, and choose any arbitrary s_n such that $n > N_s$, then by the Triangle Inequality, $\varepsilon > d(s_n, x) + d(t_k, s_n) \geq d(t_k, x)$. Observe that because the s_n term vanishes, this implies the existence of $(t_k) \in S$ such that $(t_k) \to x$. Thus, $x \in \bar{S}$.

The reverse direction is more straightforward. If $x \in \bar{S}$, then $\exists (s_n) \in S, (s_n) \to x$. However, because $S \subseteq \bar{S}$, $(s_n) \in S \Rightarrow (s_n) \in \bar{S} \Rightarrow x \in \bar{\bar{S}}$. □

Problem 5. *Prove that $\bar{S}$ is the intersection of all closed subsets in X that contain S.*

Proof. We want to show that for a metric space (X, d) and $\bar{S} \in X$,

$$\bar{S} = \bigcap \{\bar{U} : S \subseteq \bar{U} \subseteq X\}.$$

We first prove the forward direction. If $x \in \bar{S}$, suppose that $x \notin \bigcap\{\bar{U} : S \subseteq \bar{U} \subseteq X\}$. This implies $\exists \bar{U} \supseteq S$ such that $x \notin \bar{U} \Rightarrow x \in X \backslash \bar{U}$. Thus, $\bar{S} \subseteq X \backslash \bar{U} \Rightarrow S \subseteq X \backslash \bar{U}$. However, because $S \subseteq \bar{U}$ by definition, this is only possible if $S = \varnothing$. It is easy to see that if $S = \varnothing$ our result trivially holds. Thus, by contradiction, $x \in \bigcap\{\bar{U} : S \subseteq \bar{U} \subseteq X\}$. This implies $\bar{S} \subseteq \bigcap\{\bar{U} : S \subseteq \bar{U} \subseteq X\}$.

The reverse direction is more straightforward. If $x \in \bigcap\{\bar{U} : S \subseteq \bar{U} \subseteq X\}$, then $x \in \bar{U} \forall S \subseteq \bar{U} \subseteq X$. Because $S \subseteq \bar{S}$, we have $x \in \bar{S}$. This implies $\bigcap\{\bar{U} : S \subseteq \bar{U} \subseteq X\} \subseteq \bar{S}$. □