MATH 104 HW #5

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Problem 1. *Prove that* $[0,1]^2$ *in* $\mathbb{R}^2$ *is sequentially compact.*

Proof. We first prove a more general result which will directly imply the result.

Lemma 1.1. *Suppose* $U \subseteq X$ *and* $V \subseteq Y$ *are sequentially compact. Then* $U \times V \subset X \times Y$ *is also sequentially compact.*

Proof. For any $(t_n) \in U \times V$, let us write $t_n = (u_n, v_n)$. Because U is sequentially compact, then $\forall (u_n) \in U, \exists (u_{n_k}), u \in U$ such that $(u_{n_k}) \to u$. Because V is also sequentially compact and $(v_{n_k}) \in V$, $\exists (v_{m_k}), v \in U$ such that (m_k) is a subindexing of (n_k) and $(v_{m_k}) \to v$. Because (u_{m_k}) is a subsequence of (u_{n_k}), $(u_{m_k}) \to u$. Thus, the subsequence $(t_{m_k}) \to (u, v)$. By construction, $U \times V$ is also sequentially compact. □

Endowed with this lemma, because we know $[0,1]$ in $\mathbb{R}$ is sequentially compact, this implies that $[0,1]^2$ in $\mathbb{R}^2$ is sequentially compact. □

Problem 2. *Let* E *be the set of points* $x \in [0,1]$ *whose decimal expansion consists of only 4 and 7. Is* E *countable? Is* E *compact?*

Proof. Suppose E is countable. This implies that there exists a bijection $f : \mathbb{N} \to E$. Hence, we can define an enumeration (e_n) of the elements of E. For each e_i, let $e_i[j]$ denote the jth element of the decimal expansion of e_i. We use an argument similar to Cantor's diagonalization to construct an $e \in E$ as follows: if $e_i[i] = 4$ then $e[i] = 7$ and if $e_i[i] = 7$ then $e[i] = 4$. We denote this construction as the complement $e_i[i]^C$ of a digit $e_i[i]$. Because each element of e is either a 4 or a 7, $e \in E \Rightarrow e \in (e_n)$. However, $\forall e_i \exists j$ such that $e[j] \neq e_i[j] \Rightarrow e$ is not equal to any element in (e_n). This is a contradiction, so thus, E is not countable.

Now, for any $e \in E$, consider a sequence (e_n) such that for $i \leq n$, $e_n[i] = e[i]$ and $i > n$, $e_n[i] = e[i]^C$. Then,

$$
\begin{aligned}
|e_n - e| &= |\sum_{i=1}^{\infty} 10^{-i} e_n[i] - \sum_{i=1}^{\infty} 10^{-i} e[i]| = |\sum_{i=1}^{\infty} 10^{-i}(e_n[i] - e[i])| \\
&= |\sum_{i=n+1}^{\infty} 10^{-i}(e_n[i] - e[i])| \leq \sum_{i=n+1}^{\infty} 10^{-i}|e_n[i] - e[i]| \\
&\leq \sum_{i=n+1}^{\infty} 3 \cdot 10^{-i} = 10^{-n}/3 < 10^{-n}.
\end{aligned}
$$

Hence, $\forall \varepsilon > 0, N > 0, -\log_1 0(\varepsilon), \forall n > N$, $|e_n - e| < \varepsilon$. Thus, by construction, we have created a sequence $(e_n) \to e$. This implies that E contains all its limit points, which implies E is closed. Because E is bounded from below by 0 and bounded from above by 1, this implies that $E \in \mathbb{R}$ is compact. □

Problem 3. *Let $A_1, A_2, \cdots$ be subsets of a metric space. If $B = \cup_i A_i$, then $\bar{B} \supseteq \cup_i \bar{A}_j$. Is it possible that this inclusion is a strict inclusion?*

Proof. If i is finite, then we claim a strict inclusion is not possible. For any $b \in \bar{B}$, $\exists (b_n) \in B$ such that $(b_n) \to b$. Because the number of A_i is finite, by the Pigeonhole Principle, $\exists A_i$ such that there are an infinite number of elements $b_n \in A_i$. This implies that $\exists (b_{n_k}) \in A_i$ such that $(b_{n_k}) \to b \Rightarrow b \in \bar{A}_i \Rightarrow b \in \cup_i A_i$. Thus, $B = \cup_i \bar{A}_j$, which proves our result.

If i is infinite, then consider $A_i = \{\frac{1}{i}\}$. Clearly, $\bar{A}_i = A_i$. This implies that $\cup_i \bar{A}_j = \cup_i A_j = B$. However, the closure of the set $B = \{\frac{1}{i} | i \geq 1\}$ includes the limit point $0 \notin B$. Thus, a strict inclusion is possible. □

Problem 4. *Find the flaw in the reasoning of and a counterexample to the claim and its proof: Every closed subset of $\mathbb{R}$ is a countable union of closed intervals. This is because every closed set is the complement of an open set, and adjacent open intervals sandwich a closed interval.*

Proof. The flaw in the logic is that while adjacent open intervals do always sandwich a closed interval, that closed interval may only consist of a single point. Thus, a closed subset of $\mathbb{R}$ may not be able to be expressed as a countable union of points. As a counterexample, consider Problem 2, where E is closed, but also is an uncountable union of points. □