MATH 104 HW #7

James Ni

Problem 1. *If X and Y are open cover compact, prove $X \times Y$ is open cover compact.*

Proof. Let $\mathcal{U} = \bigcup_{\alpha \in \mathcal{I}} U_\alpha$ be an open cover of $X \times Y$. This implies that $\forall (x, y) \in X \times Y, \exists \alpha \in \mathcal{I}$ such that $(x, y) \in U_\alpha$. Because U_α is open, we can construct an open box $(x, y) \in A_x \times B_y \subseteq U_\alpha$ such that $x \in A_x \subseteq X$ and $y \in B_y \subseteq Y$. Now consider the set $V_x = \{(x, y) | y \in Y\}$. From our open cover construction, we can deduce that $V_x \subseteq \bigcup_{y \in Y} A_x \times B_y = A_x \times \bigcup_{y \in Y} B_y$. Because $\bigcup_{y \in Y} B_y$ is an open cover of Y, and Y is open cover compact, there exists a finite subcover $\mathcal{B}$ of Y such that $V_x \subseteq A_x \times \mathcal{B}$. We take the union over all possible V_x to get $X \times Y = \bigcup_{x \in X} V_x \subseteq \bigcup_{x \in X} (A_x \times \mathcal{B}) = \bigcup_{x \in X} A_x \times \mathcal{B}$. Similarly, because $\bigcup_{x \in X} A_x$ is an open cover of X, and X is open cover compact, there exists a finite subcover $\mathcal{A}$ of X such that $X \times Y \subseteq \mathcal{A} \times \mathcal{B}$. Because there exists a U_α for each $A_x \times B_y$ in $\mathcal{A} \times \mathcal{B}$ such that $A_x \times B_y \subseteq U_\alpha$, and that there are a finite number of union elements in $\mathcal{A} \times \mathcal{B}$, this implies that there exists a finite subcover of $\mathcal{U}$. Hence, $X \times Y$ is open cover compact. □

Problem 2. *Let $f : X \to Y$ be a continuous map between metric spaces. Let $A \subset X$. Determine if the following is true:*

- *if A is open, then $f(A)$ is open,*
- *if A is closed, then $f(A)$ is closed,*
- *if A is bounded, then $f(A)$ is bounded,*
- *if A is compact, then $f(A)$ is compact,*
- *if A is connected, then $f(A)$ is connected.*

Proof. False, consider the continuous function $f : \mathbb{R} \to \mathbb{R}, f(x) = 0$. If we let A be the open interval $(0, 1)$, we see that $f(A) = \{0\}$, which is closed and hence not open.

False, consider the continuous function $f : \mathbb{R} \to \mathbb{R}, f(x) = e^x$. If we let A be the closed interval $(-\infty, 0]$, we see that $f(A) = (0, 1]$, which is neither closed nor open.

False, consider the continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+, f(x) = \frac{1}{x}$. If we let A be the bounded interval $(0, 1]$, we see that $f(A) = [1, \infty)$, which is unbounded.

True, consider any $(s_n) \in f(A)$. We see that $\forall s_n \in f(A), \exists t_n \in A$ such that $f(t_n) = s_n$. This gives us a sequence $(t_n) \in A$. Because A is compact, $\exists (t_{n_k}) \to t$. By the definition of continuous functions, we have $f(t_{n_k}) \to f(t)$. This implies that $f(A)$ is compact.

True, suppose that $f(A)$ is not connected. This implies that $f(A) = \bigcup_{S \in \mathcal{U}} S$, where S are disjoint nonempty open subsets. Taking the preimage, we get that $A = f^{-1}(\bigcup_{S \in \mathcal{U}} S) = \bigcup_{S \in \mathcal{U}} f^{-1}(S)$. By the definition of continouous functions, each set $f^{-1}(S)$ is open and nonempty. Each of these sets also must be disjoint since otherwise, it would imply that S are not disjoint. This implies A can be represented as the union of disjoint nonempty open subsets, which implies A is not connected. Hence, by contradiction, $f(A)$ must be connected. □

Problem 3. *Prove there does not exist a continuous map* $f : [0, 1] \to \mathbb{R}$ *such that* f *is surjective.*

Proof. If such a continuous map existed, then because $[0, 1]$ is compact, it would imply $f([0, 1]) = \mathbb{R}$ is compact. This is a contradiction, so such a map cannot exist. □