MATH 104 HW #8

James Ni

Problem 1. *Let* $f_n(x) = \frac{n+\sin x}{2n+\cos n^2 x}$. *Show that* f_n *converges uniformly on* $\mathbb{R}$.

Proof. We claim that $f_n(x) \to \frac{1}{2}$. Since $|\sin x| \leq 1, |\cos n^2 x| \leq 1$ and $f_n(x)$ is nonnegative, we know that $\frac{n-1}{2n+1} \leq \frac{n+\sin x}{2n+\cos n^2 x} \leq \frac{n+1}{2n-1}$. Using this, we can deduce that the quantity $f_n(x) - \frac{1}{2}$ is bounded from below by

$$\frac{n-1}{2n+1} - \frac{1}{2} = \frac{2(n-1)-(2n+1)}{2(2n+1)} = \frac{-3}{2(2n+1)}$$

and bounded from above by

$$\frac{n+1}{2n-1} - \frac{1}{2} = \frac{2(n+1)-(2n-1)}{2(2n-1)} = \frac{3}{2(2n-1)}.$$

This implies that $|f_n(x) - \frac{1}{2}| \leq \frac{3}{2(2n-1)}$. Thus, $\forall \varepsilon > 0$, we can find an $N = \frac{3+2\varepsilon}{4\varepsilon}$ such that $\forall x \in \mathbb{R}, \forall n > N$, we have $|f_n(x) - \frac{1}{2}| < \varepsilon$. $\square$

Problem 2. *Let* $f(x) = \sum_{n=1}^{\infty} a_n x^n$. *Show that the series is continuous on* $[-1, 1]$ *if* $\sum_{n=1}^{\infty} |a_n| < +\infty$. *Additionally, prove that* $\sum_{n=1}^{\infty} n^{-2} x^n$ *is continuous on* $[-1, 1]$.

Proof. Let $f_n(x) = a_n x^n$. For $x \in [-1, 1]$, we have $|f_n(x)| = |a_n||x^n| \leq |a_n|$. Because $\sum_{n=1}^{\infty} |a_n| < +\infty$, by the Weierstrass M-test, $f(x)$ converges uniformly on $[-1, 1]$. Because each partial sum $F_n(x) = \sum_{i=1}^{n}$ is a finite polynomial, each partial sum is continuous. Hence, $f(x)$ is continuous on $[-1, 1]$.

As a brief aside, to see why a finite degree polynomial is continuous on $\mathbb{R}$, we first observe that $f(x) = c$ and $g(x) = x$ are trivially continuous. By induction, $f(x) = cx^n$ is also continuous. Because every finite degree polynomial can be decomposed as the sum of such functions, every finite degree polynomial must be continuous.

Using this, because $\sum_{n=1}^{\infty} n^{-2} < +\infty$, $\sum_{n=1}^{\infty} n^{-2} x^n$ is continuous on $[-1, 1]$. $\square$

Problem 3. *Show that* $f(x) = \sum_{n=1}^{\infty} x^n$ *represents a continuous function on* $(-1, 1)$*, but the convergence is not uniform.*

Proof. Let $f_n(x) = x^n$. For $a \in (0, 1), x \in [-a, a]$, we have $|f_n(x)| = |x^n| \leq a^n$. Because $\sum_{n=1}^{\infty} a^n < +\infty$, by the Weierstrauss M-test, $f(x)$ converges uniformly on $[-a, a]$. By similar logic in Problem 2, we deduce that $f(x)$ is continuous on $[-a, a]$, and hence on $(-1, 1)$.

However, we claim that the convergence is not uniform. To see this, let $F_n(x) = \sum_{i=1}^{n} x^n$. Observe that $x = 1$ is a limit point of the set $(-1, 1)$. Thus, if $F_n \to f$ uniformly, we have that the sequence (A_n) produced by taking the limits $A_n = \lim_{x\to 1} F_n(x) = n$ is convergent. However, (A_n) clearly diverges; by contradiction, the convergence cannot be uniform. $\square$