MATH 104 HW #9

James Ni

Problem 1. *Construct a smooth function* $f : \mathbb{R} \to \mathbb{R}$ *such that* $f(x) = 0$ *for* $x \leq 0, f(x) = 1$ *for* $x \geq 1$*, and* $f(x) \in [0, 1]$ *when* $x \in (0, 1)$*.*

Proof. Let us define the function f as

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & x > 0 \end{cases}.$$

By Example 31.3 of Ross, f is smooth and nonnegative.

We claim that the function g defined by

$$g(x) = \frac{f(x)}{f(x) + f(1-x)}$$

satisfies the required properties. We first verify the behavior of g.

If $x \leq 0$, then $g(x) = \frac{0}{0+e^{-1/(1-x)}} = 0$. If $x \geq 1$, then $g(x) = \frac{e^{-1/x}}{e^{-1/x}+0} = 1$. Because f is nonnegative, we also know that $x \in (0, 1) \Rightarrow f(x) \in [0, 1]$.

Because both the numerator and denominator are smooth, and the denominator is nonzero, g is also smooth. □

Problem 2. *Rudin 5.4. If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, \ldots, C_n$ *are real constants, prove that the equation*

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Let us define the function f as

$$f(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1}$$

such that $f(0) = f(1) = 1$. By Rolle's Theorem, there exists $x \in [0, 1]$ such that $f'(x) = 0 \Rightarrow$

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0.$$

□

Problem 3. *Rudin 5.8. Suppose f' is continuous on $[a,b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that*

$$|\frac{f(t)-f(x)}{t-x} - f'(x)| < \varepsilon$$

whenever $|t-x| < \delta$ and $x, t \in [a,b]$. This can be expressed by saying that f is uniformly differentiable on $[a,b]$ if f' is continuous on $[a,b]$. Does this hold for vector-valued functions too?

Proof. By the limit definition of f' at t, $\forall \varepsilon > 0 \exists \delta_1 > 0$ such that $\forall |t-x| < \delta$ and $x, t \in [a,b]$,

$$|\frac{f(t)-f(x)}{t-x} - f'(t)| < \varepsilon/2.$$

Because f' is continuous on a compact set by Heine-Borel, f' is uniformly continuous. Thus, $\forall \varepsilon > 0 \exists \delta_2 > 0$ such that $\forall |t-x| < \delta$ and $x, t \in [a,b]$, $|f'(t) - f'(x)| < \varepsilon/2$. If we take $\delta = \min(\delta_1, \delta_2)$, then we get

$$|\frac{f(t)-f(x)}{t-x} - f'(x)| \leq |\frac{f(t)-f(x)}{t-x} - f'(t)| + |f'(t) - f'(x)| < \varepsilon.$$

□

Problem 4. *Rudin 5.18. Suppose f is a real function on $[a,b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a,b]$. Let α, β, and P be as in Taylor's theorem. Define*

$$Q(t) = \frac{f(t)-f(\beta)}{t-\beta}$$

for $t \in [a,b], t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t-\beta)Q(t)$$

$n-1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n.$$

Proof. We first show that $Q^{(k)} : [a,b] \backslash \beta \to \mathbb{R}$ exists for $k = 1, \ldots, n-1$. We claim that

$$Q^{(k)}(t) = \frac{f^{(k)}(t) - kQ^{(k-1)}(t)}{t-\beta},$$

where $P_i(t)$ are finite degree polynomials. We shall prove this using induction on k. The base case $k = 1$ is quite simple:

$$Q'(t) = \frac{d}{dt}\frac{f(t)-f(\beta)}{t-\beta} = \frac{f'(t)(t-\beta) - (f(t)-f(\beta))}{(t-\beta)^2} = \frac{f'(t) - Q(t)}{t-\beta}.$$

Because f' exists and the denominator is nonzero, this implies Q' exists.

Now suppose that for some $k < n-1, Q^{(k)}(t)$ exists and is of the form stated above. Then,

$$\begin{aligned} Q^{(k+1)}(t) &= \frac{d}{dt}\frac{f^{(k)}(t) - kQ^{(k-1)}(t)}{t-\beta} \\ &= \frac{(f^{(k+1)}(t) - kQ^{(k)}(t))(t-\beta) - (f^{(k)}(t) - kQ^{(k-1)})}{(t-\beta)^2} \\ &= \frac{f^{(k+1)}(t) - (k+1)Q^{(k)}(t)}{t-\beta}. \end{aligned}$$

Because $f^{(k+1)}$ exists and the denominator is nonzero, this implies $Q^{(k+1)}$ exists.

Now that we have established the existence of $Q^{(k)}$, we can see that $f^{(k)}(t) = kQ^{(k-1)}(t) + (t-\beta)Q^{(k)}(t)$ for $k = 1, \dots, n-1$. Substituting into the definition of P and evaluating at β we get

$$\begin{aligned} P(\alpha) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k \\ &= f(\alpha) + \sum_{k=1}^{n-1} \frac{kQ^{(k-1)}(\alpha) + (t-\beta)Q^{(k)}(\alpha)}{k!}(\beta-\alpha)^k \\ &= f(\alpha) + \sum_{k=1}^{n-1} \frac{Q^{(k-1)}(\alpha)}{(k-1)!}(\beta-\alpha)^k - \sum_{k=1}^{n-1} \frac{Q^{(k)}(\alpha)}{k!}(\beta-\alpha)^{k+1} \\ &= f(\alpha) + Q(\alpha)(\beta-\alpha) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n \\ &= f(\beta) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n. \end{aligned}$$

This yields the alternative version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n.$$

□

Problem 5. *Rudin 5.22. Suppose f is a real function on $(-\infty, \infty)$. Call x a fixed point of f if $f(x) = x$.*
(a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.
(b) Show that the function f defined by

$$f(t) = t + (1+e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.
(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t,

prove that a fixed point of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real nuimber and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \ldots$.
(d) Show that the process described in (c) can be visualized by the zig-zag path $(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \cdots$.

Proof. Let $f(x) = x$ be a fixed point of f. Suppose that $\exists y \neq x$ such that $f(y) = y$. Without loss of generality, suppose $y > x$. Because f is differentiable, by the Mean Value Theorem there exists $t \in [x, y]$ such that $f'(t) = \frac{f(y)-f(x)}{y-x} = 1$. This is a contradiction; hence, f has at most one fixed point. □

Proof. Suppose that $\exists x$ such that $f(x) = x$. This implies $x + (1 + e^x)^{-1} = x \Rightarrow (1 + e^x)^{-1} = 0$. However, no such $x \in \mathbb{R}$ satisfies this equation. This is a contradiction; hence, f has no fixed points. Despite this, observe that $f'(t) = 1 - e^t(1 + e^t)^{-2}$. Because $e^t(1 + e^t)^{-2} > 0$ and $(1 + e^t)^2 > e^t, 0 < f'(t) < 1$. □

Proof. By the Mean Value Theorem, $\exists c \in [x_n, x_{n+1}]$ such that

$$f'(c) = \frac{f(x_{n+1} - f(x_n)}{x_{n+1} - x_n} = \frac{f(x_{n+1} - x_{n+1}}{f(x_n) - x_n}.$$

Because $|f'(c)| < 1$, this implies that the sequence $(f(x_n) - x_n) \to 0$. Hence, the point $x = \lim x_n$ satisfies $f(x) = x$, which implies x is a fixed point. □

Proof. Note that the sequence defined alternates between points on the graphs $y = x$ and $y = f(x)$. By our Mean Value Theorem construction, the path traced by these points gradually converges to the intersection of these graphs, $x = f(x)$, which defines a fixed point. □