MATH 104 HW #10

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Problem 1. *Ross 33.4. Give an example of a function f on $[0,1]$ that is not integrable for which $|f|$ is integrable.*

Proof. Consider $f : [0,1] \to \mathbb{R}$ such that $f(x) = -1$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \in \mathbb{R}\backslash\mathbb{Q}$. Then, for any partition P of $[0,1]$, we have $U(f,P) = \sum_{k=1}^{n}(t_k - t_{k-1}) = b - a$. On the other hand, by the Denseness of $\mathbb{Q}$, we have $L(f,P) = \sum_{k=1}^{n} -(t_k - t_{k-1}) = a - b$. This implies $U(f) \neq L(f)$, so thus, f is not integrable.

However, $|f(x)| = 1$ for any $x \in [0,1]$. Thus, $U(|f|,P) = L(|f|,P) = b - a$. This implies $U(|f|) = L(|f|)$ so thus, $|f|$ is integrable. □

Problem 2. *Ross 33.7. Let f be a bounded function on $[a,b]$, so that there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a,b]$.*
(a) Show $U(f^2,P) - L(f^2,P) \leq 2B[U(f,P) - L(f,P)]$ for all partitions P of $[a,b]$.
(b) Show that if f is integrable on $[a,b]$, then f^2 is also integrable on $[a,b]$.

Proof. For some partition P of $[a,b]$, let $x_k, y_k \in [t_{k-1}, t_k]$ such that $f(x_k) = M(f,P)$ and $f(y_k) = m(f,P)$. Then,

$$\begin{aligned}
U(f^2,P) - L(f^2,P) &= \sum_{k=1}^{n}(f(x_k)^2 - f(y_k)^2)(t_k - t_{k-1}) \\
&= \sum_{k=1}^{n}(f(x_k) + f(y_k))(f(x_k) - f(y_k))(t_k - t_{k-1}) \\
&\leq 2B\sum_{k=1}^{n}(f(x_k) - f(y_k))(t_k - t_{k-1}) = \\
&\leq 2B[U(f,P) - L(f,P)].
\end{aligned}$$

Thus, if f is integrable on $[a,b]$, then $\forall\varepsilon > 0 \exists P$ such that

$$U(f,P) - L(f,P) < \varepsilon/2B \Rightarrow U(f^2,P) - L(f^2,P) < \varepsilon.$$

This implies that f^2 is integrable. □

Problem 3. *Ross 33.13. Suppose f and g are continuous functions on $[a,b]$ such that $\int_a^b f = \int_a^b g$. Prove there exists x in (a,b) such that $f(x) = g(x)$.*

Proof. Let $h = f - g$ such that $\int_a^b h = 0$. By the Intermediete Value Theorem for Integrals, this implies that there exists an $x \in (a,b)$ such that $h(x) = 0 \Rightarrow f(x) = g(x)$. $\square$

Problem 4. *Ross 35.4. Let $F(t) = \sin t$ for $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Calculate*
(a) $\int_0^{\pi/2} x dF(x)$,
(b) $\int_{-\pi/2}^{\pi/2} x dF(x)$.

Proof. Because $F(t) = \sin t$ is continuously differentiable on $[-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$\begin{aligned}\int_0^{\pi/2} x dF(x) &= \int_0^{\pi/2} x \cos x dx = x \sin x|_0^{\pi/2} + \int_0^{\pi/2} \sin x dx \\ &= \frac{\pi}{2} + \cos x|_0^{\pi/2} = \frac{\pi}{2} - 1. \\ \int_{-\pi/2}^{\pi/2} x dF(x) &= \int_{-\pi/2}^{\pi/2} x \cos x dx = 0 \text{ by symmetry.}\end{aligned}$$

$\square$

Problem 5. *Ross 35.9.a). Let f be continuous on $[a,b]$. Show $\int_a^b f dF = f(x)[F(b) - F(a)]$ for some x in $[a,b]$.*

Proof. Because f is continuous on $[a,b]$, it is also F-integrable on $[a,b]$. This implies that $\int_a^b f dF = U_F(f) = L_F(f)$. By definition of sup and inf, respectively, we have $L_F(f,P) \le \int_a^b f dF \le U_F(f,P)$, for any partition of $[a,b]$. Let $M = M(f,[a,b])$ and $m = m(f,[a,b])$. Then,

$$\begin{aligned}U_F(f,P) &= \sum_{k=0}^{n} f(t_k)[F(t_k^+) - F(t_k^-)] + \sum_{k=1}^{n} M(f,(t_{k-1},t_k))[F(t_k^-) - F(t_{k-1}^+)] \\ &\le \sum_{k=0}^{n} M[F(t_k^+) - F(t_k^-)] + \sum_{k=1}^{n} M[F(t_k^-) - F(t_{k-1}^+)] = \\ &\le M[F(t_n^+) - F(t_0^-)] = M[F(b) - F(a)].\end{aligned}$$

A similar process gives us $L_F(f,P) \ge m[F(b) - F(a)]$. This implies $m[F(b) - F(a)] \le \int_a^b f dF \le M[F(b) - F(a)]$. Because $m \le f \le M$, by the Intermediete Value Theorem, this implies that there exists some $x \in [a,b]$ such that $\int_a^b f dF = f(x)[F(b) - F(a)]$. $\square$