MATH 104 HW #11

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Theorem 1. *Let f be continuous on $[g(a), g(b)]$ and g be continuously differentiable on $[a, b]$. For $x \in [a, b]$, let*

$$F(x) = \int_{g(a)}^{g(x)} f(t)dt.$$

Then, F is differentiable such that $F'(x) = f \circ g(x)g'(x)$.

Proof. By change of variables, $F(x) = \int_a^x f \circ g(x)g'(x)dx$. Because $f \circ g$ and g' are continuous, the FTC II implies that $F'(x) = f \circ g(x)g'(x)$. □

Problem 1. *Ross 34.2. Calculate*
(a) $\lim_{x\to 0} \frac{1}{x} \int_0^x e^{t^2} dt$,
(b) $\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2} dt$.

Proof. Let $F(x) = \int_0^x e^{t^2} dt$. Because $F(0) = 0$, by the definition of derivative, $F'(0) = \lim_{x\to 0} \frac{f(x)}{x}$. By the FTC II, we have $F'(0) = e^{t^2}|_{t=0} = 1$.

Let $F(x) = \int_3^{3+h} e^{t^2} dt$. Similarly, we apply $F(0) = 0$ and the definition of derivative to get $F'(0) = \lim_{h\to 0} \frac{f(h)}{h}$. By a simple application of Theorem 1, we have $F'(0) = e^{(t+3)^2}|_{t=0} = e^9$. □

Problem 2. *Ross 34.5. Let f be a continuous function on $\mathbb{R}$ and define*

$$F(x) = \int_{x-1}^{x+1} f(t)dt \text{ for } x \in \mathbb{R}.$$

Show that F is differentiable on $\mathbb{R}$ and compute F'.

Proof. We can write

$$F(x) = \int_0^{x+1} f(t)dt - \int_0^{x-1} f(t)dt.$$

By a simple application of Theorem 1, we can express F as the sum of two differentiable functions, which implies F is differentiable and $F'(x) = f(x + 1) - f(x - 1)$. □

Problem 3. *Ross 34.7. Use change of variables to integrate $\int_0^1 x\sqrt{1 - x^2}dx$.*

Proof. Let $f(x) = \sqrt{x}, g(x) = 1 - x^2$. It is easy to see that f is continuous and g is continuously differentiable on $[0, 1]$, and that $g'(x) = -2x$. By change of variables,

$$\int_0^1 x\sqrt{1-x^2}dx = -\frac{1}{2}\int_1^0 \sqrt{u}du = \frac{1}{3}u^{3/2}|_{u=0}^{u=1} = \frac{1}{3}.$$

$\square$