MATH 104 HW1

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January 2022

1 Lec 1

Exercise 1.1. Ross 1.10

Proof. Observe that $4n-1 = 2n + (2n-1)$ so the sum can be rewritten as $(2n+1) + (2n+3) + \cdots + (2n + (2n-1))$
$2(1) - 1 = 1$ so for $n = 1$ $(2(1)+1) = 3 = 3(1^2)$
Assume $P(n)$ is true. Consider the sum $(2(n+1)+1) + (2(n+3)+1) + \cdots + (2(n+1) + (2(n+1)-1) = (2n+3) + (2n+5) + \cdots + (2n-1) + (2n + (2n+1)) + (2n + (2n+3))$. Using the IH, this can be rewritten as $3n^2 - (2n+1) + (4n+1) + (4n+3) = 3n^2 + 6n + 3 = 3(n+1)^2$, as desired. □

Exercise 1.2. Ross 1.12

Proof. a. $(a+b)^1 = a + b = \binom{1}{0}a + \binom{1}{1}b$
$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2 = \binom{2}{0}a^2 = \binom{2}{1}ab + \binom{2}{2}b^2$
$(a+b)^3 = a^3 + a^2b + aba + ab^2 + ba^2 + bab + b^2a + b^3 = a^3 + 3a^2b + 3b^2a + b^3 = \binom{3}{0}a^3 + \binom{3}{1}ab^2 + \binom{3}{2}ab^2 + \binom{3}{3}b^3$
b. $\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!}{k(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k)!(n-k+1)} = \frac{(k)n! + (n-k+1)n!}{(k)!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$
c. the case where $n = 1$ follows from part a
Assume $P(n)$ is true. Consider $(a+b)^{n+1}$. By the IH, this is equivalent to $(a+b)(a+b)^n = (a+b)(\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n) = \binom{n}{0}a^{n+1} + (\binom{n}{0} + \binom{n}{1})a^n b + (\binom{n}{1} + \binom{n}{2})a^{n-1}b^2 + \cdots + (\binom{n}{n-1} + \binom{n}{n})ab^n + \binom{n}{n}b^n$.
Since $\binom{n}{0} = 1 = \binom{n+1}{0}$ and $\binom{n}{n} = 1 = \binom{n+1}{n+1}$, this is equivalent to
$\binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^n b + \binom{n+1}{2}a^{n-2}b + \cdots + \binom{n+1}{n}ab^n + \binom{n+1}{n+1}b^{n+1}$ □

Exercise 1.3. Ross 2.1

Proof. Observe that for any positive integer c, $\sqrt{c}$ solves the equation $x^2 - c = 0$ so we can apply the rational roots test to this polynomial to determine if c is rational.
$\sqrt{3}$ solves $x^2 - 3 = 0$ so the possible rational roots are $\pm 1, \pm 3$. Since $1 = \sqrt{1} < \sqrt{3} < \sqrt{4} = 2$, none of these roots work so $\sqrt{3}$ is irrational.
Similarly since $5, 7$, and 31 are all prime, using the bounds $2 = \sqrt{4} < \sqrt{5} <$

$\sqrt{3} = 9$, $2 = \sqrt{4} < \sqrt{7} < \sqrt{3} = 9$, and $5 = \sqrt{25} < \sqrt{31} < \sqrt{36} = 6$ we can see that $\sqrt{5}, \sqrt{7}$, and $\sqrt{31}$ are irrational.
For $\sqrt{24}$, observe that $\sqrt{24} = 2\sqrt{6}$ so it suffices to show $\sqrt{6}$ is irrational. The polynomial equation $x^2 - 6 = 0$ has possible roots $\pm 1, \pm 2, \pm 3, \pm 6$. Observing $2 = \sqrt{4} < \sqrt{6} < \sqrt{9} = 3$, gives the desired result. $\square$

Exercise 1.4. Ross 2.2

Proof. Since 2, 5, and 13 are all prime, using the inequalities $1 = \sqrt[3]{1} < \sqrt[3]{2} < \sqrt[3]{8} = 2$, $1 = \sqrt[7]{1} < \sqrt[7]{5} < \sqrt[7]{128} = 2$, $1 = \sqrt[4]{1} < \sqrt[4]{13} < \sqrt[4]{16} = 2$ and similar reasoning as 2.1 we can conclude all are irrational. $\square$

Exercise 1.5. Ross 2.7

Proof. a. Suppose $\sqrt{4 + 2\sqrt{3}} - \sqrt{3} = x$. Rearranging yields this yields $x^2 + 2\sqrt{3}x - 1 - 2\sqrt{3} = 0$ which implies the sum is rational. So if we assume x is rational, x^2 is rational so we can subtract out $x^2 - 1$ and still have a rational number. Thus $2\sqrt{3}(x-1)$ is rational. Since we assumed x to be rational, $x - 1$ is rational. So since $2\sqrt{3}$ is irrational in order for the product to be rational we must have $2\sqrt{3}(x-1) = 0$ so $x = 1$. Plugging this in, we see it satisfies the original relation so $x = 1$.
b. Suppose $\sqrt{6 + 4\sqrt{2}} - \sqrt{2} = x$. Rearranging yields this yields $x^2 + 2\sqrt{2}x - 4 - 4\sqrt{2} = 0$. Applying similar reasoning as part a we see $2\sqrt{2}x - 4\sqrt{2} = 2\sqrt{2}(x-2)$ is rational so $x = 2$. $\square$

Exercise 1.6. Ross Theorem 3.1

Proof. (i). If $a + c = b + c$, then adding $(-c)$ to both sides yields $a + c + (-c) = b + c + (-c)$ so $a + 0 = b + 0$ so $a = b$
(ii). Observe $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$ so by part (i) $a \cdot 0 = 0$
(iii). Observe that $(-a)b + ab = (-a + a)b = (0)b = 0$ so $(-a)b$ is an additive inverse of ab. Since inverses are unique it follows that $(-ab) = -ab$.
(iv). By part (iii), $(-a)(-b) = -(-ab)$ which is that additive inverse of $-ab$. Since ab is also an additive inverse of $-ab$, it follows $-(-ab) = ab$.
(v). If $ac = bc$ with $c \neq 0$ then c has a multiplicative inverse c^{-1}. Multiplying both sides by c^{-1} yields $acc^{-1} = bcc^{-1}$ so $a(1) = b(1)$ so $a = b$.
(vi). Suppose $ab = 0$ and assume WLOG $a \neq 0$. Observe that $ab = 0 = a(0)$ so by part (v), it follows that $b = 0$. $\square$

Exercise 1.7. Ross Theorem 3.2

Proof. (i). If $a \leq b$ observe that $a + (-a - b) \leq b + (-a - b)$ so $-b \leq -a$
(ii). If $a \leq b$ and $c \leq 0$, then by part (i), $-c \geq 0$ so $-ac \leq -bc$ so by part (i) $bc \leq ac$
(iii). Follows since $0b = 0 \leq ab$ by axiom.
(iv). Either $0 \leq a$ or $a \leq 0$. If $0 \leq a$, then by part (iii) $0 \leq a^2$. If $a \leq 0$ then $0 \leq -a$ so $0 \leq (-a)(-a) = a^2$.
(v). Observe that $1 = 1^2$ so by part (iv), $0 \leq 1^2$. Also by definition of a field,

since 0 and 1 are distinct we have $0 \neq 1$ so $0 < 1$.
(vi). If $0 < a$ observe that $0 < a^{-1}a^{-1}$ since $a^{-1} \neq 0$ for any a. So multiplying both sides by a^{-2} yields $0a^{-2} = 0 < aa^{-2} = a^{-1}$.
(vii). If $0 < a < b$ observe that $a, b \neq 0$ so they have defined inverses with $0 < a^{-1}, b^{-1}$ by part (vi) so $0 < a^{-1}b^{-1}$. Multiplying by $a^{-1}b^{-1}$ yields $0a^{-1}b^{-1} < aa^{-1}b^{-1} < ba^{-1}b^{-1}$ so $0 < b^{-1} < a^{-1}$. □