MATH 104 HW10

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Exercise 1.1 (Ross 33.4)**.** Give an example of a function f on $[0,1]$ that is not integrable for which $|f|$ is integrable.

Proof. Consider the function $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ -1 & x \notin \mathbb{Q} \cap [0,1] \end{cases}$. Observe that for all partitions P, $U(P,f) = 1$ but $L(P,f) = -1$ so the function is not integrable. Now, observe that $|f(x)| = 1$ for all $x \in [0,1]$ so $|f|$ is integrable and $U(P,f) = L(P,f) = 1$. □

Exercise 1.2 (Ross 33.7)**.** Let f be a bounded function on $[a,b]$, so that there exists $B > 0$ such that $|f(x)| < B$ for all $x \in [a,b]$

(a) Show

$$U(f^2,P) - L(f^2,P) \leq 2B[U(f,P) - L(f,P)]$$

for all partitions P of [a,b].

(b) Show that if f is integrable on $[a,b]$, then f^2 is also integrable on $[a,b]$.

Proof.

(a) Suppose f is bounded on $[a,b]$ and $|f(x)| < B$ and let P be an arbitrary partition. First, observe that on each closed interval in the partition $[t_{i-1},t_i]$, there is some x_i, y_i such that $f^2(x_i) = \sup_{[t_{i-1},t_i]} f^2$ and $f^2(y_i) = \inf_{[t_{i-1},t_i]} f^2$. Now, since $f(x_i) + f(y_i) \leq 2B$ and

$(f(x_i) - f(y_i)) \leq \sup_{[t_{i-1},t_i]} f - \inf_{[t_{i-1},t_i]} f$, it follows that

$$\begin{aligned} U(f^2,P) - L(f^2,P) &= \sum_{i=1}^{n} (f(x_i)^2 - f(y_i)^2)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} ((f(x_i) + f(y_i))(f(x_i) - f(y_i))(t_i - t_{i-1}) \\ &\leq 2B \sum_{i=1}^{n} (f(x_i) - f(y_i))(t_i - t_{i-1}) \\ &\leq 2B (\sum_{i=1}^{n} \sup_{[t_{i-1},t_i]} f - \inf_{[t_{i-1},t_i]} f)(t_i - t_{i-1}) \\ &= 2B[U(f,P) - L(f,P)] \end{aligned}$$

as desired.

(b) Next, if f is integrable on $[a,b]$, then for arbitrary $\varepsilon > 0$ there exists a partition P such that $U(f,P) - L(f,P) \leq \frac{\varepsilon}{2B}$. Now, for this same partition, $U(f^2,P) - L(f^2,P) \leq 2B[U(f,P) - L(f,P)] \leq 2B\frac{\varepsilon}{2B} = \varepsilon$. Hence, since ε was arbitrary, it follows that $U(f^2) = L(f^2)$ so f^2 is integrable on $[a,b]$.

□

Exercise 1.3 (Ross 33.13)**.** Suppose f and g are continuous functions on $[a,b]$ such that $\int_a^b f = \int_a^b g$. Prove there exists x in (a,b) such that $f(x) = g(x)$.

Proof. First, observe that since f and g are integrable, so is $f - g$ and $\int_b^a (f - g) = 0$. Further, since f and g are continuous, $f - g$ is as well so by the intermediate value theorem for integrals, there is some x in (a,b) such that $(f-g)(x) = \frac{1}{b-a}\int_a^b (f-g) = 0$. Thus, for this same x, $f(x) - g(x) = 0$ so $f(x) = g(x)$, as desired. □

Exercise 1.4 (Ross 35.4)**.** Let $F(t) \sin t$ for $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Calculate

(a) $\int_0^{\pi/2} x dF(x)$

(b) $\int_{-\pi/2}^{\pi/2} x dF(x)$

Proof.

1. First, observe that applying integration by parts, with $F_1(x) = x$, $F_2(x) = \sin x$ we see that

$$\begin{aligned} \int_0^{\pi/2} x dF_2(x) &= \frac{\pi}{2}\sin(\frac{\pi}{2}) - 0\sin(0) - \int_0^{\pi/2} \sin x dx \\ &= \frac{\pi}{2} - 0 - 1 = \frac{\pi}{2} - 1 \end{aligned}$$

2. As above, observe that

$$\int_{-\pi/2}^{\pi/2} x dF_2(x) = \frac{\pi}{2}\sin(\frac{\pi}{2}) - \frac{-\pi}{2}\sin(\frac{-\pi}{2}) - \int_{-\pi/2}^{\pi/2} \sin x dx$$
$$= \frac{\pi}{2} - \frac{\pi}{2} - 0 = 0$$

□

Exercise 1.5 (Ross 35.9)**.** Let f by continuous on $[a,b]$. Show, $\int_a^b f dF = f(x)[F(a) - F(b)]$ for some x in $[a,b]$.

Proof. Let $M = \sup_{[a,b]} f$ and $m = \inf_{[a,b]} f$. Then for $x \in [a,b]$, $m \leq f(x) \leq M$. This implies that $\int_a^b m dF \leq \int_a^b f dF \leq \int_a^b M dF$ so $m[F(a) - F(b)] \leq \int_a^b f dF \leq M[F(a) - F(b)]$ so $m \leq \frac{1}{F(a)-F(b)} \int_a^b f dF \leq M$. Now, since f is continuous and $[a,b]$ is closed f attains both m and M at points $x_0, y_0 \in [a,b]$. Thus, by the intermediate value theorem there is some x such that $x_0 \leq x \leq y_0 \in [a,b]$ and $f(x) = \frac{1}{F(a)-F(b)} \int_a^b f dF$, or equivalently, $[F(a) - F(b)]f(x) = \int_a^b f dF$. □