# MATH 104 HW11

Jad Damaj

May 1, 2022

## 1 Hw 11

**Exercise 1.1** (Ross 34.2)**.** Calculate

(a) $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt$

(b) $\lim_{h\to 0} \frac{1}{h}\int_3^{3+h} e^{t^2}\,dt$

*Proof.*

(a) First, note that $e^{t^2}$ is integrable since $f(t) = t^2$ is integrable and $g(t) = e^t$ is continuous so $g(f(t)) = e^{t^2}$ is integrable. Let $F(x) = \int_0^x e^{t^2}\,dt$ and observe that

$$\begin{aligned} F'(0) &= \lim_{x\to 0} \frac{F(x) - F(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{\int_0^x e^{t^2}\,dt - \int_0^0 e^{t^2}\,dt}{x} \\ &= \lim_{x\to 0} \frac{\int_0^x e^{t^2}\,dt}{x} \\ &= \lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt \end{aligned}$$

Thus, since $f$ is integrable and continuous at 0, $F'(0) = f(0) = 1$ so $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt = 1$

(b) As above, we note that $e^{t^2}$ is integrable and let $F(x) = \int_3^x e^{t^2}dt$.

$$\begin{aligned} F'(3) &= \lim_{h\to 0} \frac{F(3+h) - F(3)}{h} \\ &= \lim_{h\to 0} \frac{\int_3^{3+h} e^{t^2}dt - \int_3^3 e^{t^2}dt}{h} \\ &= \lim_{h\to 0} \frac{\int_3^{3+h} e^{t^2}dt}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}dt \end{aligned}$$

Thus, since $f$ is integrable and continuous at 3, $F'(3) = f(3) = e^9$ so $\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}dt = e^9$.

□

**Exercise 1.2** (Ross 34.5)**.** Let $f$ be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_{x-1}^{x+1} f(t)dt \quad \text{for} x \in \mathbb{R}$$

Show $F$ is differentiable on $\mathbb{R}$ and compute $F'$.

*Proof.* Note that since $f$ is continuous on $\mathbb{R}$ it is integrable. Observe that

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{\int_{x-1+h}^{x+1+h} f(t)dt - \int_{x-1}^{x+1} f(t)dt}{h} \\ &= \lim_{h\to 0} \frac{\int_{x+1}^{x+1+h} f(t)dt - \int_{x-1}^{x-1+h} f(t)dt}{h} \\ &= \lim_{h\to 0} \frac{\int_{x+1}^{x+1+h} f(t)dt}{h} - \lim_{h\to 0} \frac{\int_{x-1}^{x-1+h} f(t)dt}{h} \\ &= f(x+1) - f(x-1) \end{aligned}$$

Here the last equality follows from applying the second fundamental theorem of calculus to $F_1(y) = \int_{x+1}^{y} f(t)dt$ and $F_2(y) = \int_{x-1}^{y} f(t)dt$. □

**Exercise 1.3** (Ross 34.7)**.** Use change of variables to integrate $\int_0^1 x\sqrt{1-x^2}dx$.

*Proof.* Let $u = 1 - x^2$, then $du = -2x$ so $\int_0^1 x\sqrt{1-x^2}dx = \int_1^0 -\frac{1}{2}\sqrt{u}du = \int_0^1 \frac{1}{2}\sqrt{u}du = \frac{1}{3}u^{3/2}|_0^1 = \frac{1}{3}$. □

**Exercise 1.4** (Rudin 6.15)**.** Suppose that $f$ is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x)dx = 1.$$

Prove that

$$\int_a^b xf'(x)f(x)dx = -\tfrac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x)d(x) > \tfrac{1}{4}.$$

*Proof.* First, applying integration by parts with $u(x) = x$ and $v(x) = f^2(x)$, we see that $u'(x) = 1$ and $v'(x) = 2f'(x)f(x)$ so

$$\begin{aligned}\int_a^b xf'(x)f(x)dx &= \tfrac{1}{2}\int_a^b x(2f'(x)f(x))dx \\ &= \tfrac{1}{2}\Big(b(f^2(b)) - a(f^2(a)) - \int_a^b f^2(x)dx\Big) \\ &= \tfrac{1}{2}(0 - 0 - 1) \\ &= -\tfrac{1}{2}\end{aligned}$$

For the second inequality, applying 10 (c), we see that

$$\Big|\int_a^b (f'(x))(xf(x))dx\Big| \leq \Big(\int_a^b |f'(x)|^2 dx\Big)^{1/2} \cdot \Big(\int_a^b |xf(x)|^2 dx\Big)^{1/2}$$

so

$$|-\tfrac{1}{2}| \leq \Big(\int_a^b [f'(x)]^2 dx\Big)^{1/2} \cdot \Big(\int_a^b x^2 f^2(x)dx\Big)^{1/2}$$

so squaring both sides, we see

$$\tfrac{1}{4} \leq \Big(\int_a^b [f'(x)]^2 dx\Big) \cdot \Big(\int_a^b x^2 f^2(x)dx\Big)$$

Now, we note that by 10(a), the inequality is an equality iff $(f'(x))^2 = (xf(x))^2$. So to show the inequality is strict, we suppose for contradiction it is an equality and consider two cases:

**Case 1** - $0 \notin [a, b]$: First note that $f(x)$ must be nonzero at some point in $c \in (a, b)$ since the its integral is nonzero. Let $a'$ and $b'$ be such that $a' < c < b'$, $f(a') = f(b') = 0$, and $f$ has no other zeros between $a'$ and $b'$. Then, applying the mean value theorem on $[a', b']$, there must be some $c'$ such that $f'(c') = 0$ but then since $(f'(x))^2 = (xf(x))^2$ and $x \neq 0$ we must have $f(x) = 0$ contradicting our choice of $a'$ and $b'$. Hence, there can be no such $a'$ and $b'$ which implies $f$ is the zero function, contradicting our assumption.

**Case 2-** $0 \in [a, b]$: By the above discussion, since $f(a) = f(b) = 0$, we can assume WLOG $f$ has no positive zeros other than $b$ and no positive zeros other than $a$ since if it did the function would be identically zero between them. Also, we not that in order to satisfy the mean value theorem $f'(0) = 0$. Further, $x = 0$ is the only critical point of $f$, otherwise $f$ would have another 0.
Now, we can assume $f$ has a maximum at 0 (the case for minimum is analogous). This implies that $f(x) > 0$ for $x \in (a, b)$. Since $f$ has no other critical points, it is strictly decreasing on $(0, b]$ and strictly increasing on $[a, 0)$. So on $(0, b)$ we see that $f'(x) = -xf(x)$ so we can consider $f(x) - f'(x) = f(x) + xf(x) = (1 + x)f(x)$. This equals zero only when $x - 1$ or $f(x) = 0$. Now since $f(0) > 0$ (since it has a maximum at 0 and is not the zero function), we see that $f(x) - f'(x) > 0$ on $(0, b)$ which implies $|f(x)| > |f'(x)|$. Similarly, on $(a, 0)$ using $f'(x) = xf(x)$ and $f(x) - f'(x) = f(x) - xf(x)$ we see that $|f(x)| > |f'(x)|$. Finally, since either $a < 0$ or $b > 0$, there exists a point $x \in (-1, 1)$ such that $x \neq 0$. By above, at this $x$, $|f(x)| > |f'(x)|$ and since $|x| < 1$ it follows $|f(x)| > |xf'(x)| = |f(x)|$ which is a contradiction.
Thus, in both cases we get a contradiction so it cannot be an equality. □

**Exercise 1.5** (Rudin 6.16)**.** For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Prove that

(a) $\zeta(s) = s \int_1^\infty \frac{[x]}{x^{s+1}} dx$

and that

(b) $\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{x-[x]}{x^{s+1}} dx$,

where $[x]$ denotes the greatest integer $\leq x$.
Prove that the integral in $(b)$ converges for all $s > 0$.

*Proof.*

(a) To show that $\zeta(s) = s \int_1^\infty \frac{[x]}{x^{s+1}} dx$ we consider $ss \int_1^N \frac{[x]}{x^{s+1}} dx$ for integers $N$ as $N \to \infty$. Observe that

$$\begin{aligned} s \int_1^N \frac{[x]}{x^{s+1}} dx &= s \int_1^2 \frac{1}{x^{s+1}} dx + s \int_2^3 \frac{2}{x^{s+1}} dx + \cdots + s \int_{N-1}^N \frac{N-1}{x^{s+1}} dx\Big) \\ &= 1(\frac{1}{1^s} - \frac{1}{2^s}) + 2(\frac{1}{2^s} - \frac{1}{3^s}) + \cdots + (N-1)(\frac{1}{(N-1)^s} - \frac{1}{N^s}) \\ &= \sum_{n=1}^{N-1} \frac{1}{n^s} - (N-1)\frac{1}{N^s} \\ &= \sum_{n=1}^{N} \frac{1}{n^s} - \frac{1}{N^{s-1}} \end{aligned}$$

This implies that

$$\Big|\sum_{n=1}^{N}\frac{1}{n^s}-s\int_1^N\frac{[x]}{x^{s+1}}dx\Big|=\sum_{n=N+1}^{\infty}\frac{1}{n^s}+\frac{1}{N^{s-1}}$$

As $N\to\infty$, both of these terms approach 0 which gives the desired equality.

(b) Next, observe that

$$\begin{aligned}\frac{s}{s-1}-s\int_1^\infty\frac{x-[x]}{x^{s+1}}dx &= \frac{s}{s-1}-s\int_1^\infty\frac{x}{x^{s+1}}+s\int_1^\infty\frac{[x]}{x^{s+1}}dx\\ &=\frac{s}{s-1}-s\int_1^\infty\frac{1}{x^s}+\zeta(s)\\ &=\frac{s}{s-1}-s\int_1^\infty\frac{1}{x^s}+\zeta(s)\\ &=\frac{s}{s-1}-\frac{s}{s-1}+\zeta(s)\\ &=\zeta(s)\end{aligned}$$

The above integral converges for all $s>0$ since $0\le x-[x]\le 1$ so $0\le\int_1^\infty\frac{x-[x]}{x^{s+1}}dx\le\int_1^\infty\frac{1}{x^{s+1}}$ and $\int_1^\infty\frac{1}{x^{s+1}}$ converges.

□

**Exercise 1.6.** Let $f:[0,1]\to\mathbb{R}$ be given by

$$f(x)=\begin{cases}0 & \text{if } x=0\\ \sin(1/x) & \text{if } x\in(0,1]\end{cases}.$$

And let $\alpha:[0,1]\to\mathbb{R}$ be given by

$$\alpha(x)=\begin{cases}0 & \text{if } x=0\\ \sum_{n\in\mathbb{N},1/n<x}2^{-n} & \text{if } x\in(0,1]\end{cases}.$$

Prove that $f$ is integrable with respect to $\alpha$ on $[0,1]$.

*Proof.* Observe that since $f$ is bounded, and has finitely many discontinuities, it suffices to show that $\alpha(x)$ is continuous when $f$ is not. Since $f$ is only discontinuous at $x=0$, we will show $\alpha(x)$ is continuous at $x=0$.
Since $\alpha(x)=0$, for $\varepsilon>0$ we will show there is a $\delta>0$ such that if $y\in[0,1]$ and $|x-y|<\delta$, $|\alpha(y)|<\varepsilon$.
Let $\varepsilon>0$ be arbitrary. First, note that since $\sum_{i=n}^\infty 2^{-i}=(\frac{1}{2})^{n+1}$, there is some $N$ such that for all $m>N$, $\sum_{i=m}^\infty 2^{-i}=(\frac{1}{2})^{m+1}<\varepsilon$. So taking $\delta=\frac{1}{N}$, we see that if $|x-y|<\delta$, $y\in[0,\delta)$ so if $\frac{1}{m}<y$ for $m\in\mathbb{N}$ we have $\frac{1}{m}<\frac{1}{N}$ so $N<m$ so the sum of all such $m$ is $(\frac{1}{2})^{m+1}<\varepsilon$ by choice of $N$. Thus, $\alpha(x)$ is continuous at 0, as desired. □