MATH 104 HW1

Jad Damaj

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1 Hw 1

Exercise 1.1. Ross 1.10

Proof. Observe that $4n-1 = 2n + (2n-1)$ so the sum can be rewritten as $(2n+1) + (2n+3) + \cdots + (2n + (2n-1))$
$2(1) - 1 = 1$ so for $n = 1$ $(2(1)+1) = 3 = 3(1^2)$
Assume $P(n)$ is true. Consider the sum $(2(n+1)+1) + (2(n+3)+1) + \cdots + (2(n+1) + (2(n+1)-1) = (2n+3) + (2n+5) + \cdots + (2n-1) + (2n + (2n+1)) + (2n + (2n+3))$. Using the IH, this can be rewritten as $3n^2 - (2n+1) + (4n+1) + (4n+3) = 3n^2 + 6n + 3 = 3(n+1)^2$, as desired. □

Exercise 1.2. Ross 1.12

Proof. a. $(a+b)^1 = a + b = \binom{1}{0}a + \binom{1}{1}b$
$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2 = \binom{2}{0}a^2 = \binom{2}{1}ab + \binom{2}{2}b^2$
$(a+b)^3 = a^3 + a^2b + aba + ab^2 + ba^2 + bab + b^2a + b^3 = a^3 + 3a^2b + 3b^2a + b^3 = \binom{3}{0}a^3 + \binom{3}{1}ab^2 + \binom{3}{2}ab^2 + \binom{3}{3}b^3$
b. $\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!}{k(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k)!(n-k+1)} = \frac{(k)n! + (n-k+1)n!}{(k)!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$
c. the case where $n = 1$ follows from part a
Assume $P(n)$ is true. Consider $(a+b)^{n+1}$. By the IH, this is equivalent to $(a+b)(a+b)^n = (a+b)(\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n) = \binom{n}{0}a^{n+1} + (\binom{n}{0} + \binom{n}{1})a^nb + (\binom{n}{1} + \binom{n}{2})a^{n-1}b^2 + \cdots + (\binom{n}{n-1} + \binom{n}{n})ab^n + \binom{n}{n}b^n$.
Since $\binom{n}{0} = 1 = \binom{n+1}{0}$ and $\binom{n}{n} = 1 = \binom{n+1}{n+1}$, this is equivalent to
$\binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^nb + \binom{n+1}{2}a^{n-2}b + \cdots + \binom{n+1}{n}ab^n + \binom{n+1}{n+1}b^{n+1}$ □

Exercise 1.3. Ross 2.1

Proof. Observe that for any positive integer c, $\sqrt{c}$ solves the equation $x^2 - c = 0$ so we can apply the rational roots test to this polynomial to determine if c is rational.
$\sqrt{3}$ solves $x^2 - 3 = 0$ so the possible rational roots are $\pm1, \pm3$. Since $1 = \sqrt{1} < \sqrt{3} < \sqrt{4} = 2$, none of these roots work so $\sqrt{3}$ is irrational.
Similarly since $5, 7$, and 31 are all prime, using the bounds $2 = \sqrt{4} < \sqrt{5} <$

$\sqrt{3} = 9$, $2 = \sqrt{4} < \sqrt{7} < \sqrt{3} = 9$, and $5 = \sqrt{25} < \sqrt{31} < \sqrt{36} = 6$ we can see that $\sqrt{5}, \sqrt{7}$, and $\sqrt{31}$ are irrational.
For $\sqrt{24}$, observe that $\sqrt{24} = 2\sqrt{6}$ so it suffices to show $\sqrt{6}$ is irrational. The polynomial equation $x^2 - 6 = 0$ has possible roots $\pm 1, \pm 2, \pm 3, \pm 6$. Observing $2 = \sqrt{4} < \sqrt{6} < \sqrt{9} = 3$, gives the desired result. □

Exercise 1.4. Ross 2.2

Proof. Using the inequalities $1 = \sqrt[3]{1} < \sqrt[3]{2} < \sqrt[3]{8} = 2$, $1 = \sqrt[7]{1} < \sqrt[7]{5} < \sqrt[7]{128} = 2$, $1 = \sqrt[4]{1} < \sqrt[4]{13} < \sqrt[4]{16} = 2$ and similar reasoning as 2.1 we can conclude all are irrational. □

Exercise 1.5. Ross 2.7

Proof. a. Suppose $\sqrt{4 + 2\sqrt{3}} - \sqrt{3} = x$. Rearranging yields this yields $x^2 + 2\sqrt{3}x - 1 - 2\sqrt{3} = 0$ which implies the sum is rational. So if we assume x is rational, x^2 is rational so we can subtract out $x^2 - 1$ and still have a rational number. Thus $2\sqrt{3}(x-1)$ is rational. Since we assumed x to be rational, $x - 1$ is rational. So since $2\sqrt{3}$ is irrational in order for the product to be rational we must have $2\sqrt{3}(x-1) = 0$ so $x = 1$. Plugging this in, we see it satisfies the original relation so $x = 1$.
b. Suppose $\sqrt{6 + 4\sqrt{2}} - \sqrt{2} = x$. Rearranging yields this yields $x^2 + 2\sqrt{2}x - 4 - 4\sqrt{2} = 0$. Applying similar reasoning as part a we see $2\sqrt{2}x - 4\sqrt{2} = 2\sqrt{2}(x-2)$ is rational so $x = 2$. □

Exercise 1.6. Ross 3.6

Proof. a. Applying the triangle inequality twice we see $|a+b+c| = |(a+b)+c| \leq |a+b| + |c| \leq |a| + |b| + |c|$.
b. For $n = 1$, observe $|a_1| \leq |a_1|$ is true.
Suppose $P(n)$ is true and consider $|a_1 + \cdots + a_n + a_{n+1}|$. Applying the triangle inequality then the IH we see, $|a_1 + \cdots + a_n + a_{n+1}| = |(a_1 + \cdots + a_n) + a_{n+1}| \leq |a_1 + \cdots + a_n| + |a_{n+1}| \leq |a_1| + \cdots + |a_n| + |a_{n+1}|$, as desired. □

Exercise 1.7. Ross 4.11

Proof. For $a, b \in \mathbb{R}$, suppose there finitely many rationals $q_1, \ldots, q_n$ such that $a < q_1 < \cdots < q_n < b$. Viewing q_n as a real number we see that by the denseness of $\mathbb{Q}$, there exists a rational number q_{n+1} such that $q_n < q_{n+1} < b$. This contradicts our original assumption, thus there cannot be infinitely many rational between a and b. □

Exercise 1.8. Ross 4.14

Proof. a. It is evident $\sup A + \sup B$ is an upper bound since for $a + b \in A + B$, $a + b \leq \sup A + b \leq \sup A + \sup B$. To prove that $\sup A + \sup B$ is the supremum, it suffices to show that for each $\varepsilon > 0$ there exists $c \in A + B$ such that $\sup A + \sup B - \varepsilon < c < \sup A + \sup B$.
Let $\varepsilon > 0$ be arbitrary. By properties of sup, we can choose a' and b' such

that $\sup A - \frac{\varepsilon}{2} < a' < \sup A$ and $\sup B - \frac{\varepsilon}{2} < b' < \sup B$. Since $a' + b'$ is an element of $A + B$, we see that combining theses two inequalities yields $\sup A + \sup B - \varepsilon < a' + b' < \sup A + \sup B$.
b. It is evident $\inf A + \inf B$ is a lower bound since for $a + b \in A + B$, $a + b \geq \inf A + b \geq \inf A + \inf B$. To prove that $\inf A + \inf B$ is the infimum, it suffices to show that for each $\varepsilon > 0$ there exists $c \in A + B$ such that $\inf A + \inf B < c < \inf A + \inf B + \varepsilon$.
Let $\varepsilon > 0$ be arbitrary. By properties of inf, we can choose a' and b' such that $\inf A < a' < \inf A + \frac{\varepsilon}{2}$ and $\inf B < b' < \inf B + \frac{\varepsilon}{2}$. Since $a' + b'$ is an element of $A + B$, we see that combining theses two inequalities yields $\inf A + \inf B < a' + b' < \inf A + \inf B + \varepsilon$. □

Exercise 1.9. Ross 7.5

Proof. a. $s_n = \sqrt{n^2+1} - n = (\sqrt{n^2+1} - n)\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2}}$ so $\lim s_n = 0$
b. $s_n = \sqrt{n^2+n} - n = (\sqrt{n^2+n} - n)(\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n} = \frac{(\frac{1}{n})n}{\frac{1}{n}\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}+1}}$ so $\lim s_n = \frac{1}{2}$
c. $s_n = \sqrt{4n^2+n} - 2n = (\sqrt{4n^2+n} - 2n)\frac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n} = \frac{n}{\sqrt{4n^2+n}+2n} = \frac{(\frac{1}{n})n}{(\frac{1}{n})\sqrt{4n^2+n}+2n} = \frac{1}{\sqrt{4+\frac{1}{n}+2}}$ so $\lim s_n = \frac{1}{4}$ □