MATH 104 HW1

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Exercise 1.1 (Ross 9.9)**.** Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.

(a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.

(b) Prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

(c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$.

Proof.

(a) Let $M \in \mathbb{R}$ be arbitrary. Since $\lim s_n = +\infty$, we can choose N_1 such that if $n > N_1$, then $s_n > M$. Taking $N = \max\{N_0, N_1\}$, we see that if $n > N$, then $t_n \geq s_n > M$ so $\lim t_n = +\infty$.

(b) Let $M \in \mathbb{R}$ be arbitrary. Since $\lim t_n = -\infty$, we can choose N_1 such that if $n > N_1$, then $t_n < M$. Taking $N = \max\{N_0, N_1\}$, we see that if $n > N$, then $s_n \leq t_n < M$ so $\lim s_n = -\infty$.

(c) Let $\lim s_n = s$ and $\lim t_n = t$. Suppose for contradiction, $s > t$. Then, $s - t = \varepsilon > 0$ and we can choose N_1 such that if $n > N_1$, then $|s_n - s| \leq \frac{\varepsilon}{2}$ and N_2 such that if $n > N_2$, then $|t_n - t| \leq \frac{\varepsilon}{2}$. This implies that if we take $N = \max\{N_0, N_1, N_2\}$ and $n > N$, then

$$t - \frac{\varepsilon}{2} < t_n < t + \frac{\varepsilon}{2} = s - \frac{\varepsilon}{2} < s_n < s + \frac{\varepsilon}{2}.$$

This contradicts our assumption that $s_n \leq t_n$ for $n > N_0$ so we must have $\lim s_n \leq \lim t_n$.

□

Exercise 1.2 (Ross 9.15)**.** Show that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

Proof. First, observe that if $a = 0$, this gives a sequence of all 0s which converges to 0.
Suppose $a \neq 0$. If $a > 0$, then by the Archimedean principle we can choose a'

such that a' is an integer and $a' \geq a$. Also note that for all n, $0 < \frac{a^n}{n!} \leq \frac{(a')^n}{n!}$ so to show $\lim \frac{a^n}{n!} = 0$, it suffices to show $\lim \frac{(a')^n}{n!} = 0$.
Observe that for $n > 2a'$,

$$\begin{aligned}
\frac{(a')^n}{n!} &= \frac{(a')^{2a'}(a')^{n-2a'}}{(2a')!(2a'+1)(2a'+2)\cdots(2a'+(n-2a'))} \\
&\leq \frac{(a')^{2a'}(a')^{n-2a'}}{(2a')!(2a')(2a')\cdots(2a')} \\
&= \frac{(a')^{2a'}}{(2a')!2^{n-2a'}} = \frac{(a')^{2a'}2^{2a'}}{(2a')!} \cdot \frac{1}{2^n}
\end{aligned}$$

Let $\varepsilon > 0$, be arbitrary. Since $\lim \frac{1}{2^n} = 0$, we can choose N_1 such that if $n > N_1$, then

$$\frac{1}{2^n} \leq \varepsilon \cdot \frac{(2a')!}{(2a')^{2a'}} \iff \frac{(2a')^{2a'}}{(2a')!} \cdot \frac{1}{2^n} < \varepsilon.$$

Taking $N = \max\{2a, N_1\}$, we see that if $n > N$, $\frac{(a')^n}{n!} \leq \frac{(2a')^{2a'}}{(2a')!} \cdot \frac{1}{2^n} < \varepsilon$ so $\lim \frac{(a')^n}{n!} = 0$.
Similarly, if $a < 0$, we see that $-a > 0$ so by the above reasoning $\lim \frac{(-a)^n}{n!} = 0$ so $\lim -\frac{a^n}{n!} = 0$ as well. Since for all n, $\frac{a^n}{n!} = \frac{(-a)^n}{n!}$ or $= -\frac{a^n}{n!}$ and both sequences converge to 0, $\frac{a^n}{n!}$ converges to 0 as well. □

Exercise 1.3 (Ross 10.7)**.** Let S be a nonempty bounded subset of $\mathbb{R}$ such that $\sup S$ is not in S. Prove there is a sequence (s_n) of points in S_n that $\lim s_n = \sup S$.

Proof. Construct the sequence as follows: Let s_i be a point in S such that $\sup S - \frac{1}{i} < s_i < \sup S$. Such a point exists by definition of $\sup S$ and since $\sup S \notin S$.
Considering the sequence (s_n) defined above, let $\varepsilon > 0$ be arbitrary. By the archimedean principle there exists an integer N such that $\frac{1}{N} < \varepsilon$. So by construction, we see that for $n > N$, $|s_n - \sup S| < \frac{1}{N} < \varepsilon$ so $\lim s_n = \sup S$. □

Exercise 1.4 (Ross 10.8)**.** Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n)$. Prove (σ_n) is an increasing sequence.

Proof. We will show $\sigma_n \leq \sigma_{n+1}$ for all n. First, observe that

$$\begin{aligned}
\sigma_n \leq \sigma_{n+1} &\iff \frac{1}{n}(s_1 + \cdots + s_n) \leq \frac{1}{n+1}(s_1 + \cdots + s_n + s_{n+1}) \\
&\iff (n+1)(s_1 + \cdots + s_n) \leq n(s_1 + \cdots + s_n + s_{n+1}) \\
&\iff s_1 + \cdots + s_n \leq n s_{n+1}
\end{aligned}$$

Here, the last equality holds since $s_{n+1} \geq s_m$ for all $m < n+1$ so $s_1 + \cdots + s_n \leq s_{n+1} + \cdots + s_{n+1} = n s_{n+1}$. Thus, $\sigma_n \leq \sigma_{n+1}$ for all n so the sequence is increasing. □

Exercise 1.5 (Ross 10.9)**.** Let $s_1 = 1$ and $s_{n+1} = (\frac{n}{n+1})s_n^2$ for $n \geq 1$.

(a) Find s_2, s_3, and s_4.

(b) Show $\lim s_n$ exists.

(c) Prove $\lim s_n = 0$.

Proof.

(a) $s_1 = 1$, $s_2 = \frac{1}{2}$, $s_3 = \frac{1}{6}$, $s_4 = \frac{1}{48}$.

(b) To show (s_n) converges it suffices to show it is monotone and bounded.
Claim: $1 \geq s_n \geq s_{n+1} \geq 0 \ \forall n$.
We will proceed by induction.
Basis Step: Observe that $P(1)$ is true since $1 \geq 1 = s_1 \geq \frac{1}{2} = s_2 \geq 0$.
Inductive Step: Assume $P(n)$ is true. Then $1 \geq s_n \geq s_{n+1} \geq 0$. Observe that since $1 \geq s_{n+1} \geq 0$, $s_{n+1} \geq s_{n+1}^2 \geq 0$. Also, since $0 < \frac{n+1}{n+2} < 1$ it follows that $0 \leq s_{n+2} = (\frac{n+1}{n+2})s_{n+1}^2 \leq s_{n+1} \leq 1$, as desired.

(c) Since $\lim s_n$ exists, let $\lim s_n = s$. Applying $\lim_{n\to\infty}$ to both sides of the equality $s_{n+1} = (\frac{n}{n+1})s_n^2$ yields, $s = s^2$ so $s = 0$ or 1. Since (s_n) is a decreasing sequence which contains terms strictly less than 1 we see that $s = 0$.

□

Exercise 1.6 (Ross 1.10)**.** Let $s_1 = 1$ and $s_n = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

(a) Find s_2, s_3, and s_4.

(b) Use induction to show $s_n > \frac{1}{2}$ for all n

(c) Show (s_n) is a decreasing sequence.

(d) Show $\lim s_n$ exists and find $\lim s_n$.

Proof. (a) $s_1 = 1$, $s_2 = \frac{2}{3}$, $s_3 = \frac{5}{9}$, $s_4 = \frac{14}{27}$

(b) We will proceed by induction.
Basis Step: $P(1)$ is true since $s_1 = 1 > \frac{1}{2}$. Inductive Step: Assume $P(n)$ is true. Then $s_n > \frac{1}{2}$ so $s_n + 1 > \frac{3}{2}$ so $\frac{1}{3}(s_n + 1) > \frac{1}{2}$, as desired.

(c) We will proceed by induction.
Basis Step: $P(1)$ is true since $s_2 = \frac{1}{2} < 1 = s_1$.
Inductive Step: Assume $P(n)$ is true. Then, $s_{n+1} < s_n$ so $s_{n+1}+1 < s_n+1$ so $s_{n+2} = \frac{1}{3}(s_{n+1} + 1) < \frac{1}{3}(s_n + 1) = s_{n+1}$, as desired.

(d) By parts (b) and (c), (s_n) is monotone and bounded so it converges. Suppose $\lim s_n = s$. Applying $\lim_{n\to\infty}$ to both sides of the equality $s_{n+1} = \frac{1}{3}(s_n + 1)$ yields $s = \frac{1}{3}(s + 1)$ so $s = \frac{1}{2}$.

□

Exercise 1.7 (Ross 1.11). Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{4n^2}] \cdot t_n$ for $n \geq 1$.

(a) Show $\lim t_n$ exists.

Proof. (a) To show $\lim t_n$ exists it suffices to show t_n is decreasing and bounded. First, observe that $0 < 1 - \frac{1}{4n^2} < 1$ for all n so $t_n > [1 - \frac{1}{4n^2}]t_n = t_{n+1}$ for all n.
Now, we claim $t_n > 0$ for all n. To show this we will proceed by induction.
Basis Step: $P(1)$ is true since $t_1 = 1 > 0$
Inductive Step: Assume $P(n)$ is true then. $t_n > 0$. Since $0 < t - \frac{1}{4n^2}$, $t_{n+1} = [1 - \frac{1}{4n^2}]t_n > 0$.

□

Exercise 1.8 (Squeeze Test). Let a_n, b_n, c_n be three sequences such that $a_n \leq b_n \leq b_n$, and $L = \lim a_n = \lim c_n$. Show that $\lim b_n = L$.

Proof. Let $\varepsilon > 0$ be arbitrary. Since a_n and c_n converge to L we can choose N_1 such that if $n > N_1$ then $|a_n - L| < \varepsilon$, and N_2 such that if $n > N_2$ then $|c_n - L| < \varepsilon$. Taking $N = \max\{N_1, N_2\}$ we see that if $n > N$,

$$L - \varepsilon < a_n < L + \varepsilon$$

and

$$L - \varepsilon < c_n < L + \varepsilon.$$

Combining these inequalities and our assumption we see that

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$$

so $|b_n - L| < \varepsilon$ for $n > N$ so $\lim b_n = L$. □