# MATH 104 HW3

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## 1 Hw 3

**Exercise 1.1** (Ross 10.6)**.**

(a) Let $(s_n)$ be a sequence such that

$$|s_{n+1} - s_n| < s^{-n} \text{ for all } n \in \mathbb{N}.$$

Prove $(s_n)$ is a cauchy sequence and hence a convergent sequence.

(b) Is the result in $(a)$ true if we only asssume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$.

*Proof.*

(a) Note that $1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = \frac{2^{n+1}-1}{2^n}$ and $\sum_{i=0}^{\infty} \frac{1}{2^i} = 2$. Let $m, n \in \mathbb{N}$ be arbitrary and assume $m > n$. Observe that by repeated applications of the triangle inequality we have

$$\begin{aligned} |s_m - s_n| &\leq |s_m - s_{m-1}| + \cdots + |s_{n+1} - s_n| \\ &< 2^{-m} + \cdots + 2^{-n} \\ &< 2 - \frac{2^{n+1} - 1}{2^n} = \frac{1}{2^n} \end{aligned}$$

So for $\varepsilon > 0$, taking $N$ such that $\frac{1}{2^N} \leq \varepsilon$ we see that $|s_m - s_n| < \varepsilon$ for all $n, m > N$. Thus the sequence is cauchy.

(b) The result in part $(a)$ is not necessarily true if we use the bound $\frac{1}{n}$. This is because the sum $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge so summing up arbitrary many terms of the sequence is not bounded. This means we can construct a sequence that satisfies these requirements yet oscillates between 0 and $\frac{1}{2}$:

$$0, \frac{1}{2}, \frac{1}{6}, \frac{-1}{12}, \frac{7}{60}, \frac{17}{60}, \ldots$$

This sequence can continue to oscillate in this fashion since summing finitely many terms of the form $\frac{1}{n}$ can always accumulate to $\frac{1}{2}$.

□

**Exercise 1.2** (Ross 11.2)**.** Consider the sequences defined as follows:

$$a_n = (-1)^n, \quad b_n = \frac{1}{n}, \quad c_n = n^2, \quad d_n = \frac{6n+4}{7n-3}.$$

(a) For each sequence, give an example of a monotone subsequence.

(b) For each sequence, give its set of subsequential limits.

(c) For each sequence, give its lim sup and lim inf.

(d) Which if the sequences converges? diverges to $+\infty$? diverges to $-\infty$?

(e) Which of the sequences is bounded?

*Proof.*

(a) $(a_{n_k}) = 1, 1, 1, 1, \ldots$ (even terms)
$b_n$ itself is monotone decreasing
$c_n$ itself is monotone increasing
$d_n$ itself if monotone decreasing

(b) $S_a = \{1, -1\}$, $S_b = \{0\}$, $S_c = \{+\infty\}$, $S_d = \{\frac{6}{7}\}$

(c) $\limsup a_n = 1$, $\liminf a_n = -1$
$\limsup b_n = \liminf b_n = 0$
$\limsup c_n = \liminf c_n = +\infty$
$\limsup d_n = \liminf d_n = \frac{6}{7}$

(d) $b_n$ and $d_n$ converge. $c_n$ diverges to $+\infty$.

(e) $a_n$, $b_n$, and $c_n$ are bounded.

□

**Exercise 1.3** (Ross 11.3)**.** Repeat Exercise 11.2 for the sequences:

$$s_n = \cos(\frac{n\pi}{3}), \quad t_n = \frac{3}{4n+1}, \quad u_n = (-\frac{1}{2})^n, \quad v_n = (-1)^n + \frac{1}{n}.$$

*Proof.*

(a) $(s_{n_k}) = 1, 1, 1, 1, \ldots$ (every 6th term)
$t_n$ itself is monotone decreasing
$(u_{n_k}) = (\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$ (even terms)
$v_{n_k} = (1 + \frac{1}{2}, 1 + \frac{1}{4}, 1 + \frac{1}{6}, \ldots)$ (even terms)

(b) $S_s = \{1, \frac{1}{2}, -\frac{1}{2}, 1\}$, $S_t = \{0\}$, $S_u = \{0\}$, $S_d = \{1, -1\}$

(c) $\limsup s_n = 1$, $\liminf s_n = -1$
$\limsup t_n = \liminf t_n = 0$
$\limsup u_n = \liminf u_n = 0$
$\limsup v_n = 1$, $\liminf v_n = -1$

(d) $t_n$ and $u_n$ converge.

(e) $s_n$, $t_n$, $u_n$, and $v_n$ are bounded.

□

**Exercise 1.4.** Let $(q_n)$ be an enumeration of all the rationals in the interval $(0, 1]$.

(a) Give the set of subsequential limits for $(q_n)$.

(b) Give the values of $\limsup q_n$ and $\liminf q_n$.

*Proof.*

(a) $S_q = [0, 1] \subseteq \mathbb{R}$ since $\mathbb{R}$ was constructed such that it consisted of all limits of sequences in $\mathbb{Q}$.

(b) $\limsup q_n = 1$ and $\liminf q_n = 0$.

□

**Exercise 1.5.** How would you explain 'what is limsup'? For example, you can say something about: What's the difference between limsup and sup? What is most counter-intuitive about limsup? Can you state some sentences that seems to be correct, but is actually wrong?

*Proof.*

- I would explain $\limsup$ as describing the behavior of the tail end of a sequence, specifically its least upper bound.
- The difference between limsup and sup is that sup describes a single set while limsup describes the limit of sets. This means that sup is more sensitive to small changes in the set and can be affected by outliers. This is reflected in the fact that limsup is less than any value $a$ such that there are finitely many $s_m > a$.
- A counter intuitive fact about limsup is that it doesn't need to be an upper bound for any element in the set. To see this, consider the sequence $(s_n) = \frac{1}{n}$. $\limsup s_n = 0$ but $0 < \frac{1}{n}$ for all $n \in \mathbb{N}$.
- For any $a \in \mathbb{R}$ such that $a \geq \limsup s_n$, there are only finitely many terms with $s_m > a$ is a false statement that appears to be true. This is reflected by taking $a = 0$ and considering the previous example. This statement becomes true when the inequality is strict.

□