MATH 104 HW4

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Exercise 1.1 (Ross 12.10)**.** Prove (s_n) is bounded if and only if $\limsup |s_n| < +\infty$.

Proof. First, suppose (s_n) is bounded, then there exists M such that for all n, $|s_n| < M$ so $\sup\{s_n : n > 1\} < M$. Now, since $0 \le \sup\{s_n : n > N\} \le \sup\{s_n : n > 1\}$, we see that $\limsup |s_n| < +\infty$
Now, suppose we have $\limsup |s_n| < +\infty$. Let $L = \limsup |s_n|$. Observe that by definition, there exists some N such that for all $M > N$,

$$|\sup\{|s_n| : n > M\} - \limsup(|s_n|)| < 1.$$

So since the sequence of limits is decreasing we have

$$\sup\{|s_n| : n > N + 1\} < \limsup |s_n| + 1$$

which implies that for all $n > M$, $|s_n| < \limsup |s_n| + 1$. So taking $M' = \max\{|s_1|, \dots, |s_N|, |s_{N+1}, \limsup |s_n| + 1\}$, we see that $|s_n| < M'$ for all n. ☐

Exercise 1.2 (Ross 12.12)**.** Let (s_n) be a sequence of nonnegative numbers, and for each n define $\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n)$.

(a) Show
$$\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n.$$

(b) Show that if $\lim s_n$ exists, then σ_n exists and $\lim \sigma_n = \lim s_n$.

(c) Give an example where $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.

Proof. (a) The second inequality follows from definitions so we will begin by showing the rightmost inequality.
First, we claim that for $M > N$,

$$\sup\{\sigma_n : n > M\} \le \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}$$

To see this observe that for σ_n with $n > N$,

$$\begin{aligned}\sigma_n &= \frac{1}{M}(s_1 + \cdots + s_N + s_{N+1} + \cdots + s_n)\\ &\leq \frac{1}{M}(s_1 + \cdots + s_N + \sup\{s_n : n > N\} + \cdots + \sup\{s_n : n > N\}\\ &= \frac{1}{M}(s_1 + \cdots + s_N + (M-N)\sup\{s_n : n > N\})\\ &= \frac{1}{M}(s_1 + \cdots + s_N\} + \frac{M-N}{M}\sup\{s_n : n > n\}\\ &\leq \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}\end{aligned}$$

Thus, since it is an upper bound for all σ_n with $n > M$, we see that our claim is true by definition of sup.
Now, fixing N and letting M tend to infinity we see that the term containing $\frac{1}{M}$ tends to zero so $\limsup \sigma_n \leq \sup\{n : n > N\}$. Thus, letting N tend to infinity we get the result $\limsup \sigma_n \leq \limsup s_n$.
A symmetric argument can be made to show $\liminf s_n \leq \liminf \sigma_n$ using the claim that for $M > N$,

$$\frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \inf\{s_n : n > N\} \leq \inf\{\sigma_n : n > M\}.$$

□

Exercise 1.3 (Ross 14.2)**.** Determine which of the following converge. Justify your answers.

(a) $\sum \frac{n-1}{n^2}$

(b) $\sum (-1)^n$

(c) $\sum \frac{3n}{n^3}$

(d) $\sum \frac{n^3}{3^n}$

(e) $\sum \frac{n^2}{n!}$

(f) $\sum \frac{1}{n^n}$

(g) $\sum \frac{n}{2^n}$

Proof. (a) Diverges by comparison to $\frac{1}{n}$

(b) Diverges since terms don't converge to 0.

(c) Converges by comparison to $\frac{1}{n^2}$.

(d) Converges by root test.

(e) Converges by ratio test.

(f) Converges by root test.

(g) Converges by ratio test.

□

Exercise 1.4 (Ross 14.10)**.** Find a series $\sum a_n$ that diverges by Root Test but for which the Ratio Test gives no information.

Proof. Consider the sum $\sum(\frac{1}{2})^{(-1)^n-n}$. Observe that for even terms $|\frac{a_{n+1}}{n}| = 2$ but for odd terms $|\frac{a_{n+1}}{a_n}| = \frac{1}{8}$ so ratio test is inconclusive but applying root test yields $2^{\frac{1}{n}-1}$ for even terms and $2^{-\frac{1}{n}-1}$ odd terms. Both of these converge to $(\frac{1}{2})^{-1} = 2$ so it diverges by root test. □

Exercise 1.5 (Rudin 3.6)**.** Investigate the behavior (convergence or divergence) of $\sum a_n$ if

(a) $a_n = \sqrt{n+1} - \sqrt{n}$

(b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$

(c) $(\sqrt[n]{n} + 1)^n$

(d) $\frac{1}{1+z^n}$ for complex values of z

Proof.

(a) Observe that $a_n = (\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$. So $a_n \geq \frac{1}{3\sqrt{n}}$ so it diverges by comparison test.

(b) Observe that $a_n = (\frac{\sqrt{n+1}-\sqrt{n}}{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{n\sqrt{n+1}+n\sqrt{n}}$. So $a_n \leq \frac{1}{n^{3/2}}$ so it converges by comparison test.

(c) Observe that $|(\sqrt[n]{n}-1)^n|^{1/n} = |(\sqrt[n]{n}-1)|$ so $\lim |a_n|^{1/n} = 0$ so it converges by root test.

(d) If $|z| < 1$ then this sum diverges since the term $|z|^n \to 0$ so $a_n \to 1$. If $|z| \geq 1$ then $\frac{1}{2z^n} \leq a_n \leq \frac{1}{z^n}$ so by comparison test we see the sum converges for all $|z| > 1$ and for all z on the unit circle except the point $z = 1$.

□

Exercise 1.6 (Rudin 3.7)**.** Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n}$$

if $a_n \geq 0$.

Proof. Consider the terms of the sequence a_n. Divide the sequence into two portions, terms such that $a_n < \frac{1}{n}$ and terms such that $a_n \geq \frac{1}{n}$. For terms such that $a_n < \frac{1}{n}$, $\frac{\sqrt{a_n}}{n} < \frac{1}{n^{3/2}}$. Now, we consider the terms a_n such that $a_n \geq \frac{1}{n}$ and will denote them as the subsequence a_{n_k}.
First, observe that the sum $\sum_{k=0}^{\infty} a_{n_k}$ must convergent since it is contained within the original sum, and the original sum consists of all positive terms. Also since $a_n \to 0$, there must be some M such that for $n > M$ $a_n < 1$. For such n, $\sqrt{a_n} < 1$ so $\frac{\sqrt{a_n}}{n} < \frac{1}{n}$. Finally, we know by comparison test $\sum_{k=0}^{\infty} \frac{1}{n_k}$ must converge.
Now, let $\varepsilon > 0$ be arbitrary. Since $\sum_{k=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent, there is some N_1 such that for all $p > q > N_1$ $\sum_{n=q}^{p} \frac{1}{n^{3/2}} < \frac{\varepsilon}{2}$. Similarly, there exists N_2 such that for all $p > q > N_2$ $\sum_{k=q}^{p} \frac{1}{n_k} < \frac{\varepsilon}{2}$. Thus, considering the partial sum $\sum_{n=q}^{p} a_n$ with $p > q > \max\{N_1, N_2, M\}$, we see that

$$\sum_{n=q}^{p} \frac{\sqrt{a_n}}{n} \leq \sum_{n=q}^{p} \frac{1}{n^{3/2}} + \sum_{k=q}^{p} \frac{1}{n_k} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So the series converges by the cauchy criterion. □

Exercise 1.7 (Rudin 3.9)**.** If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Proof. Consider the following cases: b_n is increasing or b_n is decreasing. Since it is bounded in both cases, it must converge to some limit B. If b_n is increasing observe that the sequence $\{B - b_n\}$ is a positive decreasing sequence. A similar statement can be mode about $\{b_n - B\}$ if b_n is decreasing.
Now, since the partial sums of $\sum a_n$ form a bounded sequence, we know the series $\sum (B - b_n) a_n$ converges if b_n is increasing. Since $\sum B a_n$ also converges we can conclude that $\sum b_n a_n$ converges. A symmetrical argument shows $\sum b_n a_n$ converges if b_n is decreasing. □

Exercise 1.8 (Rudin 3.11)**.** Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \quad \text{and} \quad \sum \frac{a_n}{1+n^2 a_n}$$

Proof.

(a) We will consider two cases: either there exists some N such that for all $n > N$, $a_n < 1$ or there exists infinitely many n with $a_n \geq 1$.
Suppose there exists some N such that for all $n > N$, $a_n < 1$. Then for $n > N$, $\frac{a_n}{1+a_n} \geq \frac{a_n}{1+1} = \frac{a_n}{2}$ so applying the comparison test we see that $\sum_{n=N+1}^{\infty} \frac{a_n}{1+a_n}$ diverges. Thus, the whole sequence diverges.
Now, suppose there exists infinitely many n with $a_n \geq 1$. Then there are infinitely many terms with $\frac{a_n}{1+a_n} \geq \frac{a_n}{a_n+a_n} = \frac{1}{2}$. Thus, $\frac{a_n}{1+a_n}$ cannot converge to 0 so the sum diverges.

(b) Observe that since the terms are positive s_n forms a decreasing sequence so

$$\begin{aligned}
\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \\
&= \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} \\
&= \frac{s_{N+k} - s_N}{s_{N+k}} \\
&= 1 - \frac{s_N}{s_{N+k}}
\end{aligned}$$

Now, since s_n is increasing we can make the term $\frac{s_N}{s_{N+k}}$ arbitrarily small by increasing k . Thus, the partial sum $\sum_{n=N+1}^{N+k} \frac{a_n}{s_n}$ can be made arbitrarily close to 1 for any N so it doesn't satisfy the cauchy condition and hence doesn't converge.

(c) Since s_n forms a decreasing sequence

$$\frac{a_n}{s_n^2} \leq \frac{a_n}{s_n(s_{n-1})} = \frac{s_n - s_{n-1}}{s_n(s_{n-1})} = \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

By the above inequality we see that

$$\begin{aligned}
\sum_{n=p}^{q} \frac{a_n}{s_n^2} &= \left(\frac{1}{s_{p-1}} - \frac{1}{s_p}\right) + \left(\frac{1}{s_p} - \frac{1}{s_{p+1}}\right) + \cdots + \left(\frac{1}{s_{q-1}} - \frac{1}{s_q}\right) \\
&= \frac{1}{s_{p-1}} - \frac{1}{s_q} \\
&< \frac{1}{s_{n-1}}
\end{aligned}$$

Thus since s_n is increasing for $\varepsilon > 0$ we can choose N such that $s_n > \frac{1}{\varepsilon}$ for $n > N$ so $\sum_{n=p}^{q} \frac{a_n}{s_n^2} \leq \varepsilon$ for $p, q > N$ so the sum in convergent.

(d) First, observe that $\frac{a_n}{1+n^2a_n} \leq \frac{a_n}{n^a_n} = \frac{1}{n^2}$ so $\sum \frac{a_n}{1+n^2a_n}$ converges by comparison test.

Now, we claim that $\sum \frac{a_n}{1+na_n}$ may converge or diverge.

Consider the sequence $a_n = \frac{1}{n}$. $\sum a_n$ diverges and $\frac{a_n}{1+na_n} = \frac{1/n}{1+1} = \frac{1}{2n}$ so $\sum \frac{a_n}{1+na_n}$ diverges.

Also, consider the sequence defined as follows $a_n = \begin{cases} \frac{1}{n^2} & \text{if } n \text{ is not a perfect square} \\ 1 & \text{otherwise} \end{cases}$.

eg. $a_n = 1, \frac{1}{2^2}, \frac{1}{3^3}, 1, \frac{1}{5^2}, \cdots$ Observe that $\sum a_n$ does not converge since $\lim a_n \neq 0$. Now consider the terms $a_n = \frac{a_n}{1+na_n} = \begin{cases} \frac{1}{n^2+na_n} & \text{if } n \text{ is not a perfect square} \\ \frac{1}{1+n} & \text{otherwise} \end{cases}$.

The terms that were of the form $a_n = 1$ can now be written as $\frac{1}{1+n}$ but since n was assumed to be a perfect square their sum can be written as $\sum \frac{1}{1+n^2}$, combining this with $\sum \frac{1}{n^2+na_n}$, which converges by our previous discussion, we see that $\sum \frac{a_n}{1+na_n}$ converges.

□