MATH 104 HW5

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March, 4 2022

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Exercise 1.1 (Ross 13.3)**.** Let B be the set of all bounded sequences $x = (x_1, x_2, \ldots)$, and define $d(x, y) = \sup\{|x_j - y_j| : j = 1, 2, \ldots\}$.

(a) Show d is a metric for B.

(b) Does $d^*(x, y) = \sum_{j=1}^{\infty} |x_j - y_j|$ define a metric for B?

Proof.

(a) First, note that since the sequences are bounded, the difference between the terms of the sequence are bounded as well hence the function d is real valued. Observe that $d(x, y) = 0$ iff $\sup\{|x_j - y_j| : j = 1, 2, \ldots\} = 0$ iff $|x_j - y_j| = 0$ for $j = 1, 2, \ldots$ iff $x_j = y_j$ for $j = 1, 2, \ldots$ iff $x = y$. Also since all terms are nonnegative, the supernum is nonnegative.
Since $|x_j - y_j| = |y_j - x_j|$, it follows that $d(x, y) = d(y, x)$. Finally, consider a third sequence $z = (z_1, z_2, \ldots)$. Observe that for all j, $|x_j - y_j| \leq |x_j - z_j| + |y_j - z_j|$ so $d(x, y) = \sup\{|x_j - y_j|\} \leq \sup\{|x_j - z_j| + |y_j - z_j|\} \leq \sup\{|x_j - z_j|\} + \sup\{|y_j - z_j|\} = d(x, z) + d(y, z)$.

(b) This distance function is not a metric since the sum of bounded sequences need not converge to a real number. To this consider the sequences $x = (1, 1, \ldots)$, $y = (0, 0, \ldots)$, and observe that $d^*(x, y) = \infty \notin \mathbb{R}$.

□

Exercise 1.2 (Ross 13.5)**.**

(a) Verify one of DerMorgan's Laws for sets:

$$\bigcap\{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcap\{U : U \in \mathcal{U}\}.$$

(b) Show that the intersection of any collection of sets is a closed set.

Proof.

(a) Observe that $x \in \bigcap\{S \setminus U : U \in \mathcal{U}\}$ iff $x \in S \setminus U$ for $U \in \mathcal{U}$ iff $x \notin U$ for $U \in \mathcal{U}$ iff $x \notin \bigcup\{U : U \in \mathcal{U}\}$ iff $x \in S \setminus \bigcap\{U : U \in \mathcal{U}\}$ so the sets are equal.

(b) Suppose $\bigcap F_\alpha$ is an intersection of closed sets. Then $S \setminus F_\alpha$ is open for each α. Observe that by part (a), $S \setminus \bigcap F_\alpha = \bigcup S \setminus F_\alpha$ which is an arbitrary union of open sets so it is open. Thus, since $S \setminus \bigcap F_\alpha$ is open, it follows that $\bigcap F_\alpha$ is closed.

□

Exercise 1.3 (Ross 13.7)**.** Show that every open set in $\mathbb{R}$ is a disjoint union of a finite or infinite sequence of open intervals.

Proof. Let U be an open set in $\mathbb{R}$. We will begin by defining an equivalence relation on U by $x \sim y$ iff there is an open interval $(a, b) \subset U$ such that $x, y \in (a, b)$.
To see that this is an equivalence relation, first observe that $x \sim x$ since U is open so there is an open ball for some $r > 0$ such that $B_r(x) \subset U$ so $x \in (x - r, x + r) \subset U$. It is symmetric since if $x \sim y$, then $y \sim x$ immediately. Finally, if $x \sim y$ and $y \sim z$ then there open intervals (a, b) and (c, d) with $x, y \in (a, b) \subset U$ and $y, z \in (c, d) \subset U$. Observe that $y < b$ and $c < y$ so $c < b$, thus we see that $x, z \in (a, b) \cup (c, d) = (\min(a, c), \max(b, d)) \subset U$ so $x \sim z$.
This equivalence relation partitions U into either a finite or infinte collection of disjoint sets U_i corresponding to the equivalence classes, with $U = \bigcup U_i$. First, observe that each U_i is open. To see this, let $p \in U_i$ be arbitrary. Since U is open there is an open ball $B_r(p) \subset U$. We claim that $B_r(p) \subset U_i$ as well. For each $x \in B_r(p)$, $p, x \in (p - r, p + r) \subset U$ so $p \sim x$ so $x \in U_i$.
Now, we claim that for each i, $U_i = (\inf U_i, \sup U_i)$. Note that $\sup U_i, \inf u_i \notin U_i$ otherwise there would be an open ball centered at the point contained in U_i which suggests that the point is not an upper or lower bound, respectively. So we see that for $p \in U_i$, $\inf U_i < p < \sup U_i$ so $p \in (\inf U_i, \sup U_i)$, so we have $U_i \subset (\inf U_i, \sup U_i)$.
Next, let $p \in (\inf U_i, \sup U_i)$ be arbitrary and let $\varepsilon_1 = \sup U_i - p$ and $\varepsilon_2 = p - \inf U_i$. Now, by properties of sup and inf we can choose $p_1 \in U_i$ with $\sup U_i - \varepsilon_1 < p_1 < \sup U_i$ and $p_2 \in U_i$ with $\inf U_i < p_2 < \inf U_i + \varepsilon_2$. Since $p_1, p_2 \in U_i$, there is an open interval contained in U_i with $p_1, p_2 \in (a, b)$. Since $a < p_2 < p < p_1 < b$, this implies $p \in (a, b)$ so $p \in U_i$, as desired.
Thus, we can conclude that $U = \bigcup(\inf U_i, \sup U_i)$. □

Exercise 1.4. For a subset S of a metric space, prove that if $S_1 = \overline{S}$ and $S_2 = \overline{S_1}$, then $S_1 = S_2$.

Proof. We will show $\overline{\overline{S}} = \overline{S}$. Note that $\overline{S} \subset \overline{\overline{S}}$ so it suffices to show $\overline{\overline{S}} \subset \overline{S}$. If $s \in \overline{\overline{S}}$, then there is a sequence s_n with $s_i \in \overline{S}$ such that $s_n \to s$. Also, for each $s_i \in S_n$, there is a sequence $(s_i)_n$ such that $(s_i)_n \to s_i$. Using Cantor's diagonal argument we can consider the sequence $(s_j)_j$ for $j = 1, 2, \ldots$ and observe that it converges to s. Thus s is the limit of a sequence of terms in S so $s \in \overline{S}$. □

Exercise 1.5. Prove that $\overline{S}$ is the intersection of all closed subsets in X that contains S.

Proof. Let $\{F_\alpha\}$ be the set of all closed subsets in X contained S. We will show $\overline{S} = \bigcap\{F_\alpha\}$. First observe that by exercise 4, $\overline{S}$ is a closed set containing S so $\overline{S} \supset \bigcap\{F_\alpha\}$. Now, we will show $\overline{S} \subset \bigcap\{F_\alpha\}$.
Let $s \in \overline{S}$ be arbitrary and suppose $s \notin \bigcap\{F_\alpha\}$. Then for some α, $s \notin F_\alpha$. Since F_α is closed, F_α^c is open and $s \in F_\alpha^c$ so there is an open ball such that $B_r(x) \subset F_\alpha^c$. Now, since $s \in \overline{S}$, there is a sequence $s_n \to s$ with $s_n \in S$. Since by assumption $S \subset F_\alpha$, there is a sequence of terms in F_α converging to s. This is a contradiction since we assumed we assumed there was an open ball centered at s contained in F_α^c meaning there can be no terms of F_α within r of s for some $r > 0$. Thus, $s \in \bigcap\{F_\alpha\}$, as desired. □