MATH 104 HW6

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1 Hw 6

Exercise 1.1. In class, we proved that $[0, 1]$ is sequentially compact, can you prove that $[0, 1]^2$ in $\mathbb{R}^2$ is sequentially compact? (In general, if metric space X and Y are sequentially compact, we can show that $X \times Y$ is sequentially compact.)

Proof. We will show the more general case. Consider two metric spaces X and Y with metrics d_X and d_Y. We can define the metric space $X \times Y$ under the metric $d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$. We claim that a sequence (x_n, y_n) converges to (x, y) in $X \times Y$ if and only if x_n converges to x in X and y_n converges to y in Y.
First, suppose (x_n, y_n) converges to (x, y) in $X \times Y$. Then for all $\varepsilon > 0$ there exists so N such that if $n > N$ $\sqrt{d_X(x_n, x)^2 + d_Y(y_n, y)^2} < \varepsilon$. Taking this same N we see that for $n > N$, $d_X(x_n, x) = \sqrt{d_X(x_n, x)^2} \leq \sqrt{d_X(x_n, x) + d_Y(y_n, y)^2} \leq \varepsilon$. The same is true for $d_Y(y_n, y)$.
Now, suppose x_n converges to x in X and y_n converges to y in Y. Let $\varepsilon > 0$ be arbitrary. Observe that there exists N_X, N_Y such that if $n > N_x$ then $d_X(x_n, x) < \frac{\varepsilon}{\sqrt{2}}$ and if $n > N_Y$ then $d_Y(y_n, y) < \frac{\varepsilon}{\sqrt{2}}$. Taking $N = \max(N_X, N_Y)$ then if $n > N$, $d((x_n, y_n), (x, y)) = \sqrt{d_X(x_n, x)^2 + d_Y(y_n, y)^2} \leq \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon$, as desired.
If X and Y are sequentially compact consider the metric space $X \times Y$. Suppose (x_n, y_n) is an arbitrary sequence in $X \times Y$. Observe that since X is compact there is some subsequence x_{n_k} that converges to some $x \in X$. Considering the sequence y_{n_k}, we see that since Y is compact, there is some subsequence $y_{n_{k_l}}$ that converges to some y in Y. Now, the subseqeunce $x_{n_{k_l}}$ converges to x as well since so by the above claim we see that $(x_n, y_n)_{k_l}$ converges to (x, y) in $X \times Y$. Hence $X \times Y$ is sequentially compact. $\square$

Exercise 1.2. Let E be the set of points $x \in [0, 1]$ whose decimal expansion consist of only 4 and 7 (e.g. 0.4747744 is allowed), is E countable? is E compact?

Proof. The set E is not countable. To see this, suppose there is an enumeration of E where the ith element in the enumeration is represented by

$0.d_{i,1}d_{i,2}\cdots d_{n,1}\cdots$ which each $d_{i,j}$ is either 4,7 or 0 (eg. extending all finite decimal representations to be infinite ones). Then consider the number $d = 0.d_1d_2\cdots$ where $d_i = \begin{cases} 4 & \text{if } d_{i,i} = 7 \\ 7 & \text{otherwise} \end{cases}$. Observe that d differs from each element in the enumeration and hence was not included contradicting our assumption.

We also claim that E is compact. Since it is a subset of $\mathbb{R}$, it suffices to show that E is closed and bounded. $E \subset [0,1]$ so it is bounded. Now, to show E is closed we will show its complement is open.

Consider an arbitrary $p \in [0,1)$ and observe that it has a decimal representation given by $0.d_1d_2d_3\cdots$. Either this decimal representation terminates, eg. there is some i such that for all $j > i$ $d_j = 0$, or it continues forever.

First, suppose n has a finite decimal representation. Then there is some i such that for all $j > i$, $d_j = 0$. Then, there must be some $k \leq i$ such that d_k isn't 4 or 7. Now, consider the smallest $k' \geq k$ such that $d_{k'} \neq 9$. Such a k' must exists the decimal terminates. Then, observe that $B_r(n)$ where $r = 10^{-k'}$ contains no element of E as all such digits maintain the first $k'-1$ digits, one of which is not 4 or 7.

Now, if n has a an infinite decimal representation. Then consider the smallest k such that d_k isn't 4 or 7. There must be such a k since n is not in E. Now, take the smallest k' such that $d_{k'} \neq 9$. Such a k' must exists the decimal terminates. Then, observe that $B_r(n)$ where $r = 10^{-k'}$ contains no element of E as all such digits maintain the first $k'-1$ digits, one of which is not 4 or 7. If there is no k' such that $d_{k'} \neq 0$ then n contains an infinite tail of 9s so it is equivalent to the terminating decimal achieved by replacing the last non-9 digit d with $d+1$ so it can be treated as above.

Thus, since each element not in E has an open ball not contained in E, E is closed and hence is compact. □

Exercise 1.3. Let $A_1, A_2, \cdots$ be subset of a metric space. If $B = \bigcup_i A_i$, then $\bar{B} \supset \bigcup_i \bar{A}_i$. Is it possible that this inclusion is an strict inclusion?

Proof. Yes, it is possible that this inclusion is strict. Consider the sets $A_i = (0, 1-(\frac{1}{2})^i)$. Observe that $B = \bigcup_i A_i = (0,1)$ so $\overline{B} = [0,1]$. But, since $\overline{A}_i = [0, 1-(\frac{1}{2})^i]$ and $\bigcup_i \overline{A}_i = [0,1)$ we see that the inclusion can be strict. □

Exercise 1.4. Last time, we showed that any open subset of $\mathbb{R}$ is a countable disjoint union of open intervals. Here is a claim and argument about closed set: *every closed subset of $\mathbb{R}$ is a countable union of closed intervals. Because every closed set is the complement of an open set, and adjacent open intervals sandwich a closed interval.* Can you see where the argument is wrong? Can you give an example of a closed set which is not a countable union of closed intervals? (here countable include countably infinite and finite)

Proof. The argument presented above has an error in that it assumes the infinite union of closed sets in closed. Since there can be an infinite number of open intervals there can be an infinite number of closed intervals sandwiched between them, whose infinite union is not necessary closed.

An example of a closed set which is not a countable union of closed intervals can be seen by taking E as in problem 2 but restricting it to contain only infinite decimals with 4 or 7.

To this that this is not a countable union, we first define an equivalence relation on E by $x \sim y$ iff there is some a, b such that $x, y \in [a, b] \subset E$.

First, we will show $\sim$ is an equivalence relation.

If $x \in E$, $x \in [x, x] \subset E$ so $x \sim x$.

If $x \sim y$ then there are a, b such that $x, y \in [a, b] \subset E$. Then, clearly $y \sim x$.

If $x \sim y$ and $y \sim z$ then there is a, b such that $x, y \in [a, b] \subset E$ and $y, z \in [c, d] \subset E$. Note that $c \leq y \leq b$ so $c \leq d$ so observe that $x, z \in [a, b] \cup [c, d] = [\min(a, c), \max(b, d] \subset E$ so $x \sim z$.

Now, we show that in E each equivalence class consists of a single point of which there are uncountably many by exercise 2, hence they form an uncountable disjoint union.

If $x = 0.d_{x,1}d_{x,2}\cdots$ and $y = 0.d_{y,1}d_{y,2}$ are two distinct points in E with $x < y$ then x and y must have a different decimal representations. Let k be the smallest integer such that $d_{x,k} \neq d_{x,y}$. Observe that $x < x + 10^{-k} < y$ but $x + 10^{-k} \notin E$ so since any closed interval containing x and y contains all points between them, there can be no such closed interval. Thus, $x \not\sim y$, as desired. $\square$