MATH 104 HW7

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Exercise 1.1. If X and Y are open cover compact, can you prove that $X \times Y$ is open cover compact?

Proof. First, we claim that for U_α an open set in $X \times Y$, the sets $(U_\alpha)_X = \{x \in X | (x, y) \in U_\alpha \text{ for some } y \in Y\}$ and $(U_\alpha)_Y = \{y \in Y | (x, y) \in U_\alpha \text{ for some } x \in X\}$ are open in X and Y, respectively.
We will show this for $(U_\alpha)_X$ as the other case is identical. Let $x \in (U_\alpha)_X$ be arbitrary. Then $\exists y \in Y$ such that $(x, y) \in U_\alpha$. Since U_α is open, $\exists r > 0$ such that $B_r((x, y)) \subset U_\alpha$. We claim that $B_r(x) \subset (U_\alpha)_X$. Suppose this is not the case, then there is some $x' \in B_r(x)$ such that $x' \notin (U_\alpha)_X$. Now, this implies that $(x', y) \notin U_\alpha$ but since $d_{X\times Y}((x, y), (x', y)) = d_X(x, x') < r$, $(x'y) \in B_r((x, y))$, contradicting our assumption. Thus $(U_\alpha)_X$ is open in X.
Now, suppose $\{U_\alpha\}$ is an open cover of $X \times Y$. For $x \in X$, define the set $S_x = \{U_\alpha | x \in (U_\alpha)_X\}$. For $y \in Y$ observe that $(x, y) \in U_\alpha$ for some $U_\alpha \in S_x$ for some α since U_α it is an open cover. Since U_α is open $\exists r_y > 0$ such that $B_{r_y}((x, y)) \subset U_\alpha$. Let $V_{x,y} = B_{\frac{r_y}{2}}((x, y))$. Observe that $\bigcup_{y\in Y}(V_{x,y})_Y$ is an open cover of Y so there exists some finite subcover $(V_{x,y_1})_Y, \ldots, (V_{x,y_n})_Y$. Now, let $V_x = \bigcap_{i=1}^n (V_{x,y_i})_X$ and note that $x \in V_x$ and it is open since it the intersection of finitely many open sets.
Next, observe that $\{V_x\}_{x\in X}$ is an open cover of X so there exists a finite subcover $V_{x_1}, \ldots, V_{x_m}$. Note that by the construction of each $V_{x,y}$ we can associate with some $U_{x,y}$. We claim that $\bigcup_{i=1}^m \bigcup_{j=1}^{n_{x_i}} U_{x_i,y_j}$ is an open cover of $X \times Y$. For $(x, y) \in X \times Y$, $x \in V_{x_i}$ for some i and there exists some j such that $y \in (V_{x_i,y_j})_Y$. We claim that $(x, y) \in U_{x_i,y_j}$. To see this observe that $d(x, y) \leq d(x, x_i) + d(x_i, y) \leq \frac{r_y}{2} + \frac{r_y}{2} = r_y$ so by construction $(x, y) \in U_{x_i,y_i}$, as desired. Thus, $X \times Y$ has a finite subcover.

□

Exercise 1.2. Let $f : X \to Y$ be a continuous map between metric spaces. Let $A \subset X$ be a subset. Decide if the followings are true or not. If true, give an argument, if false, give a counter-example.

- if A is open, then $f(A)$ is open

- if A is closed, then $f(A)$ is closed.
- if A is bounded, then $f(A)$ is bounded.
- if A is compact, then $f(A)$ is compact.
- if A is connected, then $f(A)$ is connected.

Proof.

- This is false. Consider the function $f : [0, 1] \to \mathbb{R}$ by $f(x) = 0$. This is continuous since for every open set in $\mathbb{R}$ if it contains 0 then its preimage is $[0, 1]$ which is open in $[0, 1]$. If it doesn't contain 0, then its preimage in $\emptyset$.
- This is false. Consider the function $f : (0, 1) \to [0, 1]$ by $f(x) = x$. f is continuous and $(0, 1)$ is closed in $(0, 1)$ but $f((0, 1)) = (0, 1)$ is not closed in $[0, 1]$.
- This is false. Consider the function $f : (0, 1) \to \mathbb{R}$ by $f(x) = \frac{1}{x}$. This function is continuous but and $(0, 1)$ is bounded but $f(0, 1) = (1, \infty) \in \mathbb{R}$ which is not bounded.
- This is true. Suppose $\{U_\alpha\}$ is an open cover of $f(A)$. Then, since f is continuous, $f^{-1}(U_\alpha)$ is open for all α. Now since A is compact and $\{f^{-1}(U_\alpha)\}$ is open cover of A, there exists a finite subcover $f^{-1}(U_1), \ldots, f^{-1}(U_n)$. Now, $U_1, \ldots, U_n$ is also a finite subcover of $f(A)$ so $f(A)$ is compact.
- This is true. Suppose A is connected but $f(A)$ is disconnected. Then there exists open sets G, H such that $f(A) = G \sqcup H$. Since f is continuous, $f^{-1}(G)$ and $F^{-1}(H)$ are also open. Observe that these sets are also disjoint otherwise so $A = f^{-1}(G) \sqcup f^{-1}(H)$ so A is not connected, contradicting our assumption.

□

Exercise 1.3. Prove that, there is not continuous map $f : [0, 1] \to \mathbb{R}$, such that f is surjective.

Proof. Suppose these was a continuous map $f : [0, 1] \to \mathbb{R}$ that was surjective. Then, by exercise 2, since $[0, 1]$ is compact, this would imply that $f([0, 1]) = \mathbb{R}$ is compact. Since $\mathbb{R}$ is not compact there can be no such function. □