MATH 104 HW8

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Exercise 1.1. Let $f_n(x) = \frac{n+\sin x}{2n+\cos n^2 x}$, show that f_n converges uniformly on $\mathbb{R}$.

Proof. We claim that f_n converges uniformly to $\frac{1}{2}$ on $\mathbb{R}$. To see this, first observe that since $-1 \leq \sin x, \cos x \leq 1$, we have

$$\frac{n-1}{2n+1} \leq f_n(x) \leq \frac{n+1}{2n-1}.$$

Now, since $\frac{1}{2} - \frac{n-1}{2n+1} = \frac{3}{4n+2}$ and $\frac{n+1}{2n-1} - \frac{1}{2} = \frac{3}{4n-2}$,

$$\frac{3}{4n+2} \leq |f_n(x) - \tfrac{1}{2}| \leq \frac{3}{4n-2}$$

So for $\varepsilon > 0$ taking $N = \frac{1}{4}(\frac{3}{\varepsilon} + 2)$, it follows that for $n > N$, $\frac{3}{4n-2} < \varepsilon$ so $|f_n(x) - \frac{1}{2}| < \varepsilon$, as desired. □

Exercise 1.2. Let $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Show that the series is continuous on $[-1, 1]$ if $\sum_n |a_n| < \infty$. Prove that $\sum_{n=1}^{\infty} n^{-2} x^n$ is continuous on $[-1, 1]$.

Proof. First, observe that since $x \in [-1, 1]$, $x^n \in [-1, 1]$ so $|a_n x^n| \leq |a_n|$. Now, since $\sum_{n=1}^{\infty} |a_n|$ converges, letting $M_n = |a_n|$ and applying the Weierstrass M test, we see that $f(x) = \sum_{n=1}^{\infty} a_n x^n$ converges uniformly and hence, since each $a_n x^n$ is continuous, $f(x)$ is continuous as well. □

Exercise 1.3. Show that $f(x) = \sum_n x^n$ represent a continuous function on $(-1, 1)$, but the convergence is not uniform.

Proof. First, observe that to show $f(x)$ is continuous on $(-1, 1)$ is suffices to show $f(x)$ is uniformly convergent on $[-a, a]$ for all $0 < a < 1$. To see why this is true, let $x_0 \in (-1, 1)$ be arbitrary. Then there exists some $0 < a < 1$ such that $x_0 \in (-a, a)$. Now, since $f(x)$ is uniformly convergent on $[-a, a]$ and x^n is continuous for all n, $f(x)$ is continuous on $[-a, a]$. Which implies more specifically, it is continuous at x_0 since there is some $r > 0$ such that $B_r(x_0) \subset (-a, a)$ so for any $\varepsilon > 0$, $\delta > 0$ can be found such that it satisfies the continuity condition in $[a, -a]$ so taking $\delta' = \min(\delta, r)$ satisfies the continuity

condition in $(-1,1)$.
Now, to show $f(x)$ is uniformly convergent on each $[-a,a]$ for $0<a<1$ observe that $|x^n| \leq |a^n|$ so since $\sum |a^n|$ is a convergent geometric series, applying the Weierstrass M test with $M_n = |a^n|$ shows $f(x)$ is convergent on $[-a,a]$.
Finally, we will show that $f(x)$ does not converge uniformly on $(-1,1)$. We will show that for some $\varepsilon > 0$, $\forall n$ there exists x such that $|f_n(x) - f(x)| > \varepsilon$. Let $\varepsilon > 1$ and observe that for arbitrary n, $f_n(x) = \sum_{k=1}^{n} x^n = \frac{1-x^n}{1-x}$ while $f(x) = \frac{1}{1-x}$ so $|f(x) - f_n(x)| = \frac{x^n}{1-x}$. Now, if $\frac{(1/2)^n}{1-(1/2)} > 1$ let $x = \frac{1}{2}$, otherwise assume $x > \frac{1}{2}$ so $\frac{x^n}{1-x} > \frac{(1/2)^n}{1-x}$. Choose x such that $x > 1 - (\frac{1}{2})^n$, we see that $\frac{(1/2)^n}{1-x} > 1$ so $\frac{x^n}{1-x} > 1$, as desired. $\square$