MATH 104 HW9

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Exercise 1.1. Construct a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x \geq 1$, and $f(x) \in [0, 1]$ when $x \in (0, 1)$

Proof. Consider the function given by $f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}}} & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$.

We claim that $f(x)$ satisfies the desired conditions.
First, we show that for each n,

$$f^{(n)}(x) = \sum_{j=1}^{n} \sum_{k=1}^{j} \frac{(-1)^{j-1}(e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^k}{(1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^j}(p_{j,k}(\tfrac{1}{x}) + q_{j,k}(\tfrac{1}{x-1}))$$

where each $p_{j,k}, q_{j,k}$ are polynomials with degree at least n.
For $n = 1$, observe that

$$f'(x) = \frac{-e^{\frac{1}{2x}+\frac{1}{2(x-1)}}}{1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}}}\left(\frac{-1}{2x} + \frac{-1}{2(x-1)}\right)$$

Now, suppose $f^{(n)}$ has the desired form. We will show $f^{(n+1)}$ does as well. Consider an arbitrary term of the summation

$$h(x) = \frac{(-1)^{j-1}(e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^k}{(1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^j}(p_{j,k}(\tfrac{1}{x}) + q_{j,k}(\tfrac{1}{x-1}))$$

Observe that

$$\begin{aligned} h'(x) =& \frac{(-1)^{j}(e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^{k+1}}{(1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^{j+1}}(p_{j,k}(\tfrac{1}{x}) + q_{j,k}(\tfrac{1}{x-1}))(-\tfrac{1}{2x^2} - \tfrac{1}{2(x-1)^2}) \\ &+ \frac{(-1)^{j-1}(e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^k}{(1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^j}(p_{j,k}(\tfrac{1}{x}) + q_{j,k}(\tfrac{1}{x-1}))(-\tfrac{1}{2x^2} - \tfrac{1}{2(x-1)^2}) \\ &+ \frac{(-1)^{j-1}(e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^k}{(1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^j}(p'_{j,k}(\tfrac{1}{x})(-\tfrac{1}{x^2}) + q'_{j,k}(\tfrac{1}{x-1})(-\tfrac{1}{(x-1)^2})) \end{aligned}$$

so $h'(x)$ has the desired form since in each term $p_{j,k}$'s and $q'_{j,k}s$ degree increases by at least one.
Finally, to show that the derivative is 0 at 0 and 1, we show that the limit of each term is 0 at 0 and 1. To show this we begin by observing

$$\lim_{x\to 1} \frac{(-1)^{j-1}(e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^k}{(1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^j} = \frac{\lim_{x\to 1}(-1)^{j-1}(e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^k}{\lim_{x\to 1}(1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^j} = \frac{0}{1} = 0$$

and

$$\begin{aligned}\lim_{x\to 0}\left|\frac{(-1)^{j-1}(e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^k}{(1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^j}\right| &= \lim_{x\to 0}\left|\frac{(e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^k}{(1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^j}\right| \\ &\le \lim_{x\to 0}\left|\frac{(1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^k}{(1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^j}\right| \\ &= \lim_{x\to 0}\left|\frac{1}{(1+e^{\frac{1}{2x}+\frac{1}{2(x-1)}})^{j-k}}\right| \\ &= 0\end{aligned}$$

Thus, since the exponential in $e^{\frac{1}{2x}+\frac{1}{2(x-1)}}$ grows faster than any polynomial of fixed degree as $x \to 0$ or $x \to 1$, we must have $f^{(n)}(0) = f^{(n)}(1) = 0$. Hence, the function is smooth. □

Exercise 1.2 (Rudin 5.4)**.** If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where the $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least 1 real root between 0 and 1.

Proof. Let $F(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_n}{n+1}x^{n+1}$. First, observe that $F'(x) = C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$. Also, observe that $F(0) = 0$ and $F(1) = 0$ so by the mean-value theorem, there is some $c \in (0,1)$ such that $0 = \frac{F(1)-F(0)}{1-0} = F'(c)$. Thus, the polynomial has a zero in $(0,1)$, as desired. □

Exercise 1.3 (Rudin 5.8)**.** Suppose that f' is continuous on $[a,b]$ and that $\varepsilon > 0$. Prove that there exists $\delta > 0$ such

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \varepsilon$$

whenever $0 \le |t-x| \le \delta$, $a \le x \le b$, $a \le t \le b$.

Proof. Let $\varepsilon > 0$ be arbitrary. Since f' is continuous there is some $\delta > 0$ such that if $|t - x| < \delta$, then $|f'(t) - f'(x)| < \varepsilon$. We claim that this same δ satisfies the condition. Consider $\frac{f(t)-f(x)}{t-x}$ and suppose $|t - x| < \delta$. First, since $t \in (x - \delta, x + \delta)$, by the mean value theorem there is some $t' \in (x - \delta, x + \delta)$ such that $\frac{f(t)-f(x)}{t-x} = f'(t')$. Further, since $|x - t'| < \delta$, by the continuity of $f'(x)$ and choice of δ, it follows that $|f'(t) - f'(x)| < \varepsilon$ so $\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \varepsilon$, as desired. □

Exercise 1.4 (Rudin 5.18)**.** Suppose that f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α, β, and P by as in Taylor's Theorem (Rudin Thm 5.15). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's Theorem:

$$f(\beta) = P(\beta) + \frac{Q^{n-1}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

Proof. Fix, arbitrary α, β, and f. We will proceed by induction on n.
For $n = 1$, We have

$$P(\beta) = \sum_{k=0}^{0} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k$$

and

$$Q^{(0)}(\alpha) = Q(\alpha) = \frac{f(\alpha) - f(\beta)}{\alpha - \beta}$$

so

$$\begin{aligned} P(\beta) + \frac{Q^{(0)}(\alpha)}{0!}(\beta - \alpha) &= f(\alpha) + \frac{f(\alpha) - f(\beta)}{\alpha - \beta}(\beta - \alpha) \\ &= f(\alpha) + (-1)(f(\alpha) - f(\beta)) \\ &= f(\beta) \end{aligned}$$

Now, suppose the statement holds for all $m < n$ and consider

$$f(\beta) = P(\beta) + \frac{Q^{n-1}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

Note that the $n - 1$rst derivative of $f(t) - f(\beta) = (t - \beta)Q(t)$ is

$$f^{(n-1)} = (n-1)Q^{(n-2)}(t) + (t - \beta)Q^{(n-1)}(t)$$

so we have

$$P(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k$$

and

$$Q^{(n-1)}(\alpha) = \frac{f^{(n-1)}(\alpha) - (n-1)Q^{(n-2)}(\alpha)}{\alpha - \beta}$$

so

$$P(\beta) + \frac{Q^{n-1}(\alpha)}{(n-1)!}(\beta-\alpha)^n = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k + \frac{f^{(n-1)}(\alpha) - (n-1)Q^{(n-2)}(\alpha)}{(\alpha-\beta)(n-1)!}(\beta-\alpha)^n$$

$$= \sum_{k=0}^{n-2} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n-1} + \frac{(-1)(\beta-\alpha)^{n-1}}{(n-1)!}(f^{(n-1)}(\alpha) - (n-1)Q^{(n-2)}(\alpha))$$

$$= \sum_{k=0}^{n-2} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k + \frac{Q^{(n-2)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n-1} + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n-1} - \frac{f^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n-1}$$

$$= \sum_{k=0}^{n-2} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k + \frac{Q^{(n-2)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n-1}$$

$$= f(\beta)$$

where the last equality follows from the inductive hypothesis. □

Exercise 1.5 (Rudin 5.22)**.** Suppose that f is a real differentiable function on $(-\infty, \infty)$. Call x a fixed point of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

(b) Show that the function defined by

$$f(t) = t + (1 - e^t)^{-1}$$

has no fixed point, although, $0 < f'(t) < 1$ for all real t.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$ where x_1 is an real number and

$$x_{n+1} = f(x_1)$$

for $n = 1, 2, 3, \ldots,$

Proof.

(a) Suppose $f'(t) \neq 1$ for all real t and f had more than 1 fixed point. Consider fixed points x_1, x_2 and observe that by the mean value theorem, there must be some $c \in (-\infty, \infty)$ such that $f'(c) = \frac{f(x_1)-f(x_2)}{x_1-x_2} = \frac{x_1-x_2}{x_1-x_2} = 1$, contradicting our assumption.

(b) Observe that if x is a fixed point, we must have $f(x) = x$ so $x = x + (1 - e^x)^{-1}$ so $(1 - e^x)^{-1} = 0$ which is impossible so f has no fixed points.

(c) Now, consider the sequence (x_n) as defined above. We will show the sequence is cauchy and hence convergent. First, observe that for arbitrary n, $|x_n - x_{n+1}| \leq |A|^{n-1}|x_1 - x_2|$.
To see this, observe that $\frac{|x_n - x_{n+1}|}{|x_{n-1} - x_n|} = \frac{|f(x_{n-1}) - f(x_n)|}{|x_{n-1} - x_n|} = |f'(c)| \leq A$ for some c by the mean value theorem. Hence, repeated applications of this gives the desired inequality.
Now, observe that for $n < m$

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x_{n+1}| + \cdots + |x_{m-1} - x_m| \\ &\leq |A|^n|x_1 - x_2| + \cdots + |A|^m|x_1 - x_2| \\ &\leq \frac{|A|^n}{1 - |A|}|x_1 - x_2| \end{aligned}$$

Since $|A| < 1$, $|A|^n \to 0$ so it can be made arbitrarily small so the series is Cauchy and hence convergent.
Finally, we will show $x = \lim x_n$ is a fixed point. Since f is differentiable, it must be continuous so for all $\varepsilon > 0$ there is some $\delta > 0$ such that if $|t - x| < \delta$, $|f(t) - f(x)| < \varepsilon/2$. Since $x_n \to x$, there is some N such that for all $n > N$, $|x - x_n| < \min(\delta, \varepsilon/2)$. Now, for all such n, $|x - x_n| < \delta$ so $|f(x) - f(x_n)| = |f(x) - x_{n+1}| < \varepsilon/2$. Hence, $|f(x) - x| \leq |f(x) - x_{n+1}| + |x_{n+1} - x| < \varepsilon$. Thus, since ε was arbitrary, we must have $f(x) = x$.

□