HW 11

Ross 34.2, 34.5, 34.7

Optional:

Rudin: Ex 15 (Hint: use 10(c)), 16

and an extra one:

Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sin(1/x) & \text{if } x \in (0,1] \end{cases}.$$

And let $\alpha : [0,1] \to \mathbb{R}$ be given by

$$\alpha(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sum_{n \in \mathbb{N}, 1/n < x} 2^{-n} & \text{if } x \in (0,1] \end{cases}.$$

Prove that f is integrable with respect to α on $[0,1]$. Hint: prove that $\alpha(x)$ is continuous at $x = 0$.

Ross 34.2. a) $\begin{cases} \lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2} dt \\ F(x) = \int_0^x e^{t^2} dt \end{cases}$

with $F \in C^1$,

$$\lim_{x \to 0} \frac{F(x)}{x} = \lim_{x \to 0} \frac{F'(x)}{1} = e^{x^2} \Big|_{x=0}$$

$$= e^0 = 1$$

b) $\begin{cases} \lim_{h \to 0} \frac{1}{h} \int_3^{3+h} e^{t^2} dt, \\ F(x) = \int_3^x e^{t^2} dt \end{cases}$

$$\Rightarrow \lim_{h \to 0} \frac{F(h+3) - F(3)}{h} = F'(3) = e^9$$

definition

5, Let $g(x) = \int_1^x f(t)\,dt$

$\Rightarrow f(x) = g'(x)$

$\Rightarrow$

$$F(x) = \int_{x-1}^{x+1} f(t)\,dt$$

$$= \int_1^{x+1} f(t)\,dt + \int_{x-1}^{1} f(t)\,dt$$

$$= \int_1^{x+1} f(t)\,dt - \int_1^{x-1} f(t)\,dt$$

$$= g(x+1) - g(x-1)$$

Since $f(t)$ is cont on $\mathbb{R}$

$\Rightarrow g(t)$ is cont on $\mathbb{R}$

$\Rightarrow F(x) = g(x+1) - g(x-1)$ is cont on $\mathbb{R}$.

Then proved

7. $\int_0^1 x\sqrt{1-x^2}\,dx$

let $k = 1-x^2 \Rightarrow k \in [0,1]$

$\Rightarrow dk = -2x\,dx$, $x = \sqrt{1-k}$

$\Rightarrow dx = \dfrac{dk}{-2\sqrt{1-k}}$

$\Rightarrow \int_0^1 x\sqrt{1-x^2}\,dx$

$= \int_1^0 \sqrt{1-k}\cdot\sqrt{k}\cdot\dfrac{1}{-2\sqrt{1-k}}\,dk$

$= \int_0^1 \dfrac{\sqrt{k}}{2}\,dk$

$= \dfrac{2}{3}\cdot\dfrac{k^{\frac{3}{2}}}{2}\Big|_0^1 = \dfrac{1}{3}$