HW 1 01/28/2022

Chap 1

#1.10

For $n=1$

left: $2n+1=3$

right $3n^2=3$ ✓

Assume for $n=k$ satisfies

then $\underbrace{(2k+1)+(2k+3)\ldots(4k-1)}_{k\text{ items}}=3k^2$

~~For $n=k+1$~~

~~left: $2k+3+2k+5+\ldots(4k+3)$~~

~~$=(2k+1+2)+(2k+3+2)$~~

$+(2k+2k-1)+4k+3$ ~~$+2k+2k-1+2$~~

For $n=k+1$

left: $(2k+3)+(2k+5)\ldots(4k+1)+4k+3$ ($k+1$ terms)

$=(2k+1+2)+(2k+3+2)\ldots(2k+2k-1+2)+(4k+3)$

$=2\cdot k+\underbrace{(2k+1)+\ldots+(2k+2k-1)}_{3k^2}+(4k+3)$

$=3k^2+2k+4k+3=3(k+1)^2$

right $=3(k+1)^2$

∴ proved.

#1.12

(a) For $n=1$

left: $(a+b)^1 = a+b$

right: $\binom{1}{0}a^1 + \binom{1}{1}a^0 b$

$= a+b$ ✓

for $n=2$

left: $(a+b)^2 = (a+b)(a+b) = a^2+2ab+b^2$

right: $\binom{2}{0}a^2 + \binom{2}{1}a^1 b + \binom{2}{2} b^2$

$= a^2+2ab+b^2$ ✓

For $n=3$

left: $(a+b)^3 = (a+b)^2(a+b) = (a^2+2ab+b^2)(a+b)$

$= a^3+3a^2b+3ab^2+b^3$

right: $\binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{0}b^3$

$= a^3+3a^2b+2ab^2+b^3$

(b) $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ①

$\binom{n}{k-1} = \frac{n!}{(k-1)!(n-k+1)!}$ ②

$\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!} =$ ③

① + ② $= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$

$= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)!}{k!(n-k+1)!}$

$= \frac{n!(n+1)}{k!(n-k+1)!}$

$\Rightarrow$ ① + ② = ③ Proved

(c) ~~For~~ From (a) we can see that for $n=1$ satisfy

then assume For $n=k$, binomial theorem hold.

$\Rightarrow$

$$(a+b)^k = \binom{k}{0} a^k \dots + \dots = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$$

For $n=k+1$

$$\text{left: } (a+b)^{k+1} = (a+b)(a+b)^k$$

$$= (a+b) \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$$

$$= \sum_{i=0}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^{i+1}$$

$$\text{right: } \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i$$

$$= \sum_{i=0}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^{i+1}$$

$$= \binom{k}{0} a^{k+1} b^0 + \sum_{i=1}^{k+1} a^{k+1-i} b^i$$

$$= \binom{k+1}{0} a^{k+1} b^0 + \sum_{i=1}^{k+1} a^{k+1-i} b^i$$

$$= \sum_{i=0}^{k+1} a^{k+1-i} b^i$$

$\Rightarrow$ left = right

then for $n=k+1$ hold.

$\Rightarrow$ proved.

Chap 2

#2.1 a) $\sqrt{3}$

$x^2-3=0$

$\Rightarrow c_n=1 \quad c_0=-3$

Let $r=\cancel{\frac{q}{d}}\ \frac{m}{n}$, where m, n are prime integers ~~nums~~.

$\Rightarrow m=\pm1, \pm3$

$n=\pm1$

for any combination for $\frac{m}{n}$, is not the solution of $x^2-3=0$.

$\Rightarrow$ ~~r does~~ rational number r does not exist

$\Rightarrow \sqrt{3}$ is not an rational num

b) $\sqrt{5}$.

Same as a)

$x^2-5=0$

$c_n=1, \ c_0=-5$

$\Rightarrow m=\pm1, \pm5$

$n=\pm1$.

$\Rightarrow$ no $r=\frac{m}{n}$, s.t $\left(\frac{m}{n}\right)^2-5=0$

$\Rightarrow \sqrt{5}$ is not rational number

d) $\sqrt{7}$

not an rational numbers

d) $\sqrt{24}$

$x^2 - 24 = 0$

$C_n = 1$ $C_0 = -24$

$m = \pm 1$ ± 2, ± 3, ± 4, ± 6, ± 8, ± 12, ± 24

$n = \pm 1$

there is not any $r = \frac{m}{n}$, s.t $r^2 - 24 = 0$

$\Rightarrow \sqrt{24}$ is not ratinal number

e) $\sqrt{31}$

$x^2 - 31 = 0$

~~m~~ $C_0 = -31$, $C_n = 1$

$m = \pm 1$, ± 31 ~~Cn~~ $n = \pm 1$

$r = \frac{m}{n}$ doesnot hold for $r^2 - 31 = 0$

$\Rightarrow$ ~~B~~ $\sqrt{31}$ is not ratinal number

#2.2 a) For $\sqrt[3]{2}$

$x = \sqrt[3]{2}$

$\Rightarrow x^3 - 2 = 0$

$C_n = 1$ $C_0 = -2$

Let $r = \frac{m}{n}$ (m, n have no common factors ~~are prime numbers~~.)

$\Rightarrow m = \pm 1$, ± 2

$n = \pm 1$

there is no $r = \frac{m}{n}$ s.t $r^3 - 2 = 0$

$\Rightarrow \sqrt[3]{2}$ is not rational number

b) $\sqrt[7]{5}$

$x^7 - 5 = 0$

$r = \frac{m}{n}$, $m = \pm 1, \pm 5$

$n = \pm 1$

$\Rightarrow \sqrt[7]{5}$ is not a rational number

c) $\sqrt[4]{13}$

$x^4 - 13 = 0$

$r = \frac{m}{n}$ $m = \pm 13, \pm 1$

$n = \pm 1$

$\Rightarrow \sqrt[4]{13}$ is not rational number

#2.7 1a) $\sqrt{4+2\sqrt{3}} - \sqrt{3}$

$x = \sqrt{4+2\sqrt{3}} - \sqrt{3}$

$\Rightarrow$

$[(x+\sqrt{3})^2 - 4]^2 = 8$

$\Rightarrow$

$x^4 + 4\sqrt{3}x^3 + 6x^2 - 4\sqrt{3}x - 7 = 0$ ⁎

$m = \pm 1, \pm 7$

$n = \pm 1$

Since $r = \frac{m}{n} = 1$ when $\begin{cases} m = 1 & n = 1 \\ \text{or } m = -1 & n = -1 \end{cases}$,

is a root for ⁎.

$\Rightarrow \sqrt{4+2\sqrt{3}} - \sqrt{3}$ is a rational number

b) $\sqrt{6+4\sqrt{2}} - \sqrt{2}$

$x = \sqrt{6+4\sqrt{2}} - \sqrt{2}$

$\Rightarrow x^4 + 4\sqrt{2}x^3 - 16\sqrt{2}x - 16 = 0$ ※

Let $r = \frac{m}{n}$ $m = \pm 1, \pm 2, \pm 4, \pm 8$

$n = \pm 1$

when $\begin{cases} m = 2 & n = 1 \\ m = -2 & n = -1 \end{cases} \Rightarrow r = 2$, s.t. $r = 2$ is a solution for ※

$\Rightarrow \sqrt{6+4\sqrt{2}} - \sqrt{2}$ is a rational number

Chap 3

#3, b

As triangle inequality.

(a) $|a+b+c| \leq |a+b| + |c|$

and $|a+b| \leq |a| + |b|$

$\Rightarrow |a+b+c| \leq |a| + |b| + |c|$

(b) For $n=1$

$|a_1| \leq |a_1|$ ✓

For $n=2$

$|a_1 + a_2| \leq |a_1| + |a_2|$ as triangle inequality

For $n=3$

proved in (a),

assume For $n=k$, the statement hold true
$\Rightarrow |a_1+a_2 \dots a_n| \le |a_1| + |a_2| \dots + |a_n|$

Then For $n=k+1$
~~left~~ $|a_1+a_2+a_n+a_{n+1}| = |(a_1+a_2 \dots a_n)+a_{n+1}|$

as triangle $\le |a_1+a_2 \dots a_n| + |a_{n+1}|$
inequality $\le |a_1| + |a_2| \dots + |a_n| + |a_{n+1}|$

$\Rightarrow$ Statement hold true for $n=k+1$

$\Rightarrow$ proved

Chap 4

#4.11 Assumption; there are ~~finitely~~ finitly rational ~~exactly~~ n numbers of rational number between a, b whenever $a<b$
let these rational number be $x_1, x_2 \dots x_n$.
s.t. ~~$a < x_i < b$~~ $a < x_1 < x_2 \dots < x_n < b$,
$\therefore$ $x_n < b$
as Denseness 4.7
there is a rational $r = x_{n+1}$, s.t.
$x_{n+1} < r < b$,
which is ~~an~~ a contradiction of the assumption
$\Rightarrow$ proved.

#4.14

Let $\sup A = S_A$, $\sup B = S_B$ $\sup(A+B) = S_{A+B}$

then for any $a \in A$, $a \le S_A$

for any $b \in B$, $b \le S_B$

① WTS $S_A + S_B \geq S_{A+B}$

For ~~the~~ any element $m \in A+B$

$m = a+b$ for $a \in A$, $b \in B$

$\Rightarrow m \le S_A + S_B$ for any $m \in A+B$

then $S_A + S_B \geq S_{A+B}$ ✓

② WTS $S_A + S_B \le S_{A+B}$

for each $b \in B$, $a \in A$

$a+b \le S_{A+B}$

$\Rightarrow S_{A+B} - b \geq a$ ~~for e~~

$\Rightarrow S_{A+B} - b$ is upper bound of A

$\Rightarrow S_{A+B} - b \geq S_A$

$\Rightarrow b \le S_{A+B} - S_A$

$\Rightarrow S_{A+B} - S_A$ is upper bound for B

$\Rightarrow S_{A+B} - S_A \geq S_B$

$\Rightarrow S_A + S_B \le S_{A+B}$

With ① ②

$\sup(A+B) = \sup A + \sup B$ proved

b) Prove $\inf(A+B) = \inf A + \inf B$

① ~~To st~~ WTS $\inf(A+B) \leq \inf A + \inf B$

Let for $a \in A$, $a \geq \inf A$

$b \in B$ $\quad b \geq \inf B$

for each $a \in A$, $b \in B$

$a+b \geq \inf(A+B)$

$\Rightarrow \inf(A+B) - a \leq b$

$\Rightarrow \inf(A+B) - a \leq \inf B$

$\therefore a \geq \inf(A+B) - \inf B$

$\Rightarrow \inf(A+B) - \inf B \leq \inf A$

$\Rightarrow \inf(A+B) \leq \inf A + \inf B$

② WTS $\inf(A+B) \geq \inf A + \inf B$

for any $x \in A+B$, $x = a+b$,

$\Rightarrow x \geq \inf A + \inf B$

$\Rightarrow \inf A + \inf B \leq \inf(A+B)$

Above : proved.

Chap 7

#75 (a) $\lim S_n = \sqrt{n^2+1} - n$

$= \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n^2+1}+n}$

$= 0$

(b) $\lim\limits_{n\to\infty} (\sqrt{n^2+n} - n)$

$= \dfrac{n}{\sqrt{n^2+n}+n}$

$= \dfrac{1}{\sqrt{\frac{n^2}{n}+\frac{1}{n}}+1} = \dfrac{1}{\sqrt{1+\frac{1}{n}}+1} = \dfrac{1}{2}$

(c) $\lim(\sqrt{4n^2+n} - 2n)$

$= \lim \dfrac{n}{\sqrt{4n^2+n}+2n} = \lim \dfrac{1}{\sqrt{4+\frac{1}{n}}+2}$

$= \dfrac{1}{4}$