Math 104

HW 2 02/03/2022

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9.9 (a) Since $\lim s_n = +\infty$

$\Rightarrow$ for each $M > 0$, $\exists N$, s.t

$n > N$ and $s_n > M$

and $\because s_n \le t_n$

$\Rightarrow$ $t_n > M$ for $n > N$,

$\Rightarrow$ ~~lim~~ as definition of ~~+∞~~.

$\lim t_n = +\infty$

(b) Since $\lim t_n = -\infty$, as definition

for each $M < 0$, $\exists N$, s.t

$n > N$ and $t_n < M$

~~and~~ $\because s_n \le t_n$

$\Rightarrow s_n < M$. for ~~some~~ N. and $n \ge N$

$\Rightarrow \lim s_n = -\infty$

(c)

Let $s = \lim s_n$ and $t = \lim t_n$.

~~$\Rightarrow$ lim $|s_n - s| < \varepsilon$ for $n > N$~~

~~$\exists N$,~~

Let ~~k_n =~~ $c_n = t_n - s_n$.

$\because t_n \ge s_n$ for all $n > N_0$

$\Rightarrow c_n \ge 0$ for ~~some~~ all $n > N_0$

Then WTS $\lim c_n \ge 0$ Let $\lim c_n = c$

(1) all $\varepsilon > 0$

~~c_n~~ $|c_n - c| < \varepsilon$ for $n > N_1$, $N_1 \in \mathbb{N}$

$\Rightarrow c-\varepsilon < C_n < c+\varepsilon$ for all $n > N_1$

Assume $c < 0$

$\Rightarrow c+\varepsilon < 0$ for all $n > \max\{N_0, N_1\}$

$C_n <$

which is contradict to $C_n > 0$ for all $n > N_0$.

$\Rightarrow$ Assumption fail.

$\Rightarrow c \geq 0$

~~which~~ $\Rightarrow \lim C_n \geq 0$

$\Rightarrow \lim t_n - s_n \geq 0$

$\Rightarrow$ proved

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9.15 To show $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$

$$\lim_{n\to\infty} \left| \frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}} \right| = \left| \frac{a}{n+1} \right| = 0 \quad \text{for all } a \in \mathbb{R}$$

$\Rightarrow$

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

#10.7

Let $\sup S = s$

$\Rightarrow s$ is the least upper bound for S.

and for $n \in \mathbb{N}$, $\frac{1}{n} > 0$,

$\Rightarrow s - \frac{1}{n} < s$

can construct an s_n, such that

$s - \frac{1}{n} < s_n < s + \frac{1}{n}$

$\Rightarrow |s_n - s| < \frac{1}{n}$ for all $n \in \mathbb{N}$

$\therefore \lim s_n = s$

#10.8

For ~~case~~ $n=1$, $n=2$

$\sigma_1 = s_1$, $\sigma_2 = \frac{1}{2}(s_1 + s_2)$

$\sigma_2 - \sigma_1 = \frac{s_2}{2} - \frac{s_1}{2} > 0$

~~Let~~ For $n, n+1$

$\sigma_n = \frac{1}{n}(s_1 + s_2 \cdots + s_n)$

$\sigma_{n+1} = \frac{1}{n+1}(s_1 \cdots s_{n+1})$

$\Rightarrow \sigma_{n+1} - \sigma_n = \frac{1}{n(n+1)}\left[-s_1 - s_2 \cdots - s_n + n s_{n+1}\right]$

$= \frac{1}{n(n+1)}\left[(s_{n+1} - s_1) + (s_{n+1} - s_2) \cdots (s_{n+1} - s_n)\right]$

> 0

$\Rightarrow$ proved.

#10.9 1(a) $S_2 = \frac{1}{2}$ $S_3 = \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$ $S_4 = \frac{3}{4} \times \frac{1}{36}$
$= \frac{1}{48}$

(b) $\cancel{\frac{S_{n+1} - S_n}{S_n}} = \left(\frac{n}{n+1}\right) S_n - \cancel{S_n}$

WTS S_n is bounded and decreasing
and $0 \le S_n \le 1$ for $n \geq 1$
$\Rightarrow 0 < S_{n+1} \le S_n \le 1$

Induction
for case $n=1$, and 2
$S_1 = 1$ $S_2 = \frac{1}{6}$
$\Rightarrow 0 < S_2 < S_1 \le 1$ ✓

Assume $0 < S_{n+1} < S_n \le 1$.

WTS $0 < S_{n+2} < S_{n+1} \le 1$
$S_{n+2} = \left(\frac{n+1}{n+2}\right) S_{n+1}^2$

$\Rightarrow \frac{S_{n+2}}{S_{n+1}} = \frac{n+1}{n+2} S_{n+1}$

$0 < \frac{n+1}{n+2} < 1$. $0 < S_{n+1} < 1$

$\Rightarrow 0 < \frac{S_{n+2}}{S_{n+1}} < 1$

$\Rightarrow S_{n+2} < S_{n+1} \le 1$

and clearly $S_{n+2} = \frac{n+2}{n+2} S_{n+1}^2 > 0$

$\Rightarrow 0 < S_{n+2} < S_{n+1} < 1$
proved.

$\Rightarrow S_n$ is bounded
and decreasing
$\Rightarrow \lim S_n$ exist

(c) $\because \lim S_n = \lim S_{n+1} = \lim \frac{n}{n+1} S_n^2 = \lim \frac{n}{n+1} \cdot \lim S_n^2$

Let $\lim S_n = S$ $= S^2$

$\Rightarrow S = S^2$

$\Rightarrow S = 0$ or 1.

From (b) $S_{n+1} < S_n \leq 1$

it is decreasing

$\Rightarrow \lim S_n = S < 1$

$\Rightarrow S = 0$

$\Rightarrow$ proved

#11

(0, 1)

(a) $S_1 = 1$ $S_2 = \frac{2}{3}$ $S_3 = \frac{1}{3}(1 + \frac{2}{3}) = \frac{5}{9}$

$S_4 = \frac{1}{3}(1 + \frac{5}{9}) = \frac{14}{27}$

(b) For $n = 1$ $S_1 = 1 > \frac{1}{2}$ ✓

Assume for $n = k$ $S_k > \frac{1}{2}$,

Then for $n = k+1$, $S_{k+1} = \frac{1}{3}(S_k + 1)$

$= \frac{1}{3} + \frac{S_k}{3}$

$S_{k+1} - \frac{1}{2} = \frac{S_k}{3} - \frac{1}{6}$.

$\because S_k > \frac{1}{2}$

$\Rightarrow S_{k+1} - \frac{1}{2} > 0$.

$\Rightarrow$ proved

(c) From (b) we know S_n have a lower bound $\frac{1}{2}$.

~~S_{n+1}~~

$$S_{n+1} - S_n = \frac{1}{3} - \frac{2}{3} S_n$$

$$\because S_n > \frac{1}{2}$$

$$\Rightarrow \frac{1}{3} - \frac{2}{3} S_n < 0$$

$$\Rightarrow S_{n+1} - S_n < 0$$

$\Rightarrow$ decreasing

(d) Since $S_n > \frac{1}{2}$ and S_n is decreasing.

$$\Rightarrow \frac{1}{2} < S_n \leq 1$$

$\Rightarrow$ $\lim S_n$ exist

$$\lim S_{n+1} = \lim S_n = \lim \frac{1}{3}(S_n + 1) = S$$

$$\Rightarrow \frac{1}{3}S + \frac{1}{3} = S$$

$$\Rightarrow S = \frac{1}{2}$$

$$\Rightarrow \lim S_n = \frac{1}{2}$$

#1 (a)

[0,1] To check $\{t_n\}$ is a bounded monotone sequence

① ~~$t_{n+1} > 0$~~

$\frac{t_{n+1}}{t_n} = 1 - \frac{1}{4n^2} < 1$

$\Rightarrow \{t_n\}$ is decreasing

② To check $\{t_n\} > 0$

~~$t_{n+1} - t_n =$~~

For case $n=1$ $t_1 = 1 > 0$

Assume $n=k$ $t_k > 0$

Then for $n=k+1$

$t_{k+1} = [1 - \frac{1}{4k^2}] t_k > 0$

$\Rightarrow t_n > 0$

$\Rightarrow 0 < t_n < t_1 = 1$

$\Rightarrow t_n$ is bounded

Above $\Rightarrow$ $\lim t_n$ exist

(b) ~~$\lim t_{n+1} = \lim t_n = \lim [1 - \frac{1}{4n^2}] t_n$~~

$= \lim (1 - \frac{1}{4n^2}) \lim t_n$

$= \lim t_n$

(b)

$$\frac{t_n}{t_{n-1}} = \cancel{4}\,1 - \frac{1}{4(n-1)^2} = \frac{4(n-1)^2 - 1}{4(n-1)^2}$$

$$= \frac{4n^2 - 8n + 3}{4(n-1)^2}$$

$$= \frac{(2n-1)(2n-3)}{4(n-1)^2}$$

2. squeeze test:

$\because \lim a_n = \lim c_n = L \quad \forall \varepsilon > 0 \ \exists N,$ for $n > N$

$\Rightarrow$ ~~$\forall \varepsilon > 0$ for $n > N$~~ $|a_n - L| < \varepsilon \Rightarrow L - \varepsilon \le a_n \le \varepsilon + L$

$|c_n - L| < \varepsilon \quad L - \varepsilon \le c_n \le \varepsilon + L$

$\because a_n \le b_n \le c_n$

$\therefore L - \varepsilon \le a_n \le b_n$

and $\quad b_n \le c_n \le L + \varepsilon$

$\Rightarrow L - \varepsilon \le b_n \le L + \varepsilon$

$\Rightarrow \quad |b_n - L| < \varepsilon$, for $n > N$

$\Rightarrow \quad \lim b_n = L$