HW3_104 Jie Zheng 02/09/2022

10.6

a) $\because |s_{n+1} - s_n| < 2^{-n}$ for all $n \in \mathbb{N}$

To show $\{s_n\}$ is Cauchy sequence.

for any $\varepsilon > 0$, need to find an N

with $m > n > N$, to get $|s_m - s_n| < \varepsilon$

$$|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} \ldots \ldots s_n|$$

$$\le |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| \ldots |s_{n+1} - s_n|$$

$$\le \frac{1}{2}^{m-1} + \frac{1}{2}^{m-2} \ldots + \frac{1}{2}^{n}$$

$$= \left(\frac{1}{2}\right)^n \left(1 + \frac{1}{2} + \frac{1}{4} \ldots \frac{1}{2}^{m-n-1}\right)$$

$$= \left(\frac{1}{2}\right)^n \left(\frac{1 - \left(\frac{1}{2}\right)^{m-n}}{1 - \frac{1}{2}}\right)$$

$$= \frac{\left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^m}{\frac{1}{2}}$$

$$= \frac{1}{2}^{n-1} - \left(\frac{1}{2}\right)^{m-1}$$

$$< \left(\frac{1}{2}\right)^{n-1}$$

Then we need

$$|S_m - S_n| < (\tfrac{1}{2})^{n-1} < \varepsilon$$

$$2^{1-n} < \varepsilon$$

$$\log_2 \varepsilon > 1-n$$

$$n > 1-\log_2 \varepsilon$$

for any $\varepsilon > 0$,

We ~~need~~ can find $N \geq 1-\log_2 \varepsilon$ to,

so that $|S_n - S_m| < \varepsilon$

$\Rightarrow \{S_n\}$ is cauchy sequence

$\Rightarrow \{S_n\}$ is convergent

b)

$|S_{n+1} - S_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$

$|S_m - S_n| = |S_m - S_{m-1} + S_{m-1} - S_{m-2} \ldots - S_n|$

$\leq \frac{1}{m-1} + \frac{1}{m-2} + \ldots \frac{1}{2} + 1$

Consider the sequence

$$a_n = \frac{1}{n-1}$$

with integral test

$$\lim a_n = \lim_{n \to \infty} \ln(n-1) \to \infty$$

$\Rightarrow\ !\exists N$ s.t. $m > n > N$, implied

$$|S_m - S_n| < \varepsilon$$

11.2

For $a_n = (-1)^n$

(a) subsequence can be

$$\{1, 1, 1, 1, 1 \ldots\}$$

(b) two subsequences $\{1, 1, 1, 1, \ldots 1\}$

$$\{-1, -1, \ldots\ldots -1\}$$

The the set of subsequence $\Rightarrow$ $\{1, -1\}$

c)

Let $S = \{1, -1\}$

$\Rightarrow \limsup s_n = \sup S = 1$

$\liminf s_n = \inf S = -1$

d) Since $\sup S \neq \inf S$

$\Rightarrow \{a_n\}$ diverge.

e) $-1 \leq a_n \leq 1$, bounded

For $b_n = \frac{1}{n}$

(a) $\left\{ \frac{1}{1}, \frac{1}{3}, \ldots \frac{1}{n} \right\}$ is a monotone decreasing subsequence

(b) for all subsequence,

the subsequence limit is 0.

$\Rightarrow$ set of subsequence limit is $\{0\}$

(c) $\liminf b_n = \limsup b_n = 0$.

(d) converge to 0., $\lim b_n = 0$

(e)

$1 \geq b_n > 0$

For $c_n = n^2$

(a) $\{4, 9, 16 \ldots\ldots (n-1)^2, n^2\}$

(b) $+\infty$

(c) $+\infty, +\infty$

(d) diverge to $+\infty$

(e) lower bounded but not upper bounded

For $d_n = \frac{6n+4}{7n-3}$

$$d_n = \frac{6n+4}{7n-3} = \frac{\frac{6}{7}(7n-3)+4-\frac{28}{7}}{7n-3}$$

$$= \frac{6}{7} + \frac{\frac{10}{7}}{7n-3}$$

a) $\left\{ \frac{16}{11}, \frac{22}{18} \cdots \frac{6n+4}{7n-3} \right\}$

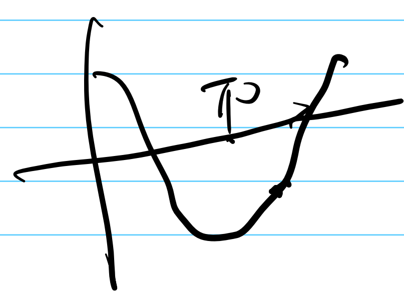
montone decreasing

b) $\frac{6}{7}$

c) $\liminf = \limsup = \lim c_n = \frac{6}{7}$

d) converge to $\frac{6}{7}$

e) $\frac{6}{7} < b_n \leq \frac{10}{4}$



11.3

For $S_n = \cos\frac{n\pi}{3} \Rightarrow \{\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1\} \ldots$

a) $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$.

b) $\{1, \frac{1}{2}, -\frac{1}{2}, -1\}$

c) $\liminf = -1$

$\limsup = 1$

d) divergent

e) $-1 \leq S_n \leq 1$

For $t_n = \frac{3}{4n+1}$

a) $\{\frac{3}{9}, \frac{3}{13}, \frac{3}{17} \ldots \frac{3}{4n+1}\}$

monotone decreasing

b) $\{0\}$

c) $\liminf = \limsup = 0$

d) converge to 0

e) $\frac{3}{5} \geq b_n > 0$

For $u_n = \left(-\frac{1}{2}\right)^n$ $\quad \left\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16} \ldots\right.$

$= (-1)^n \cdot \left(\frac{1}{2}\right)^n$

a) $\left\{ -\frac{1}{2}, -\frac{1}{8}, -\frac{1}{32} \ldots, -\left(\frac{1}{2}\right)^{2k-1} \right\}$ $k \in \mathbb{N}$

$\left\{ \frac{1}{4}, \frac{1}{16} \ldots\ldots \left(\frac{1}{2}\right)^{2k} \right\}$ $k \in \mathbb{N}$

b) $\{0\}$

c) $\liminf = \limsup = 0 = \lim u_n$

d) converge to 0

e) $-\frac{1}{2} \leq u_n \leq \frac{1}{4}$

For $v_n = (-1)^n + \frac{1}{n}$

$\left\{ 0, \frac{1}{2}, -1+\frac{1}{3}, \frac{1}{4}, \frac{-4}{5}, \ldots \right\}$

(a) $\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{6} \ldots \ldots \frac{1}{2k} \right\}$ $k \in \mathbb{N}$

Monotone decreasing

(b) $\{-1, 0\}$

c) $\liminf = -1$, $\limsup = 0$

d) diverge to $+\infty$

e) $-1 < u_n \le \frac{1}{2}$.

#11.5 (a) $S = [0,1]$

(b) $\limsup q_n = 1$

$\liminf q_n = 0$.

Discussion about lim sup.

① Let S be the set of all subsequen

~~of~~ limit of $\{s_n\}$.

then $\sup S$ is the biggest

ellment of S,

② limsup is defined as the

sup of the "very ~~ta~~ tail" of this

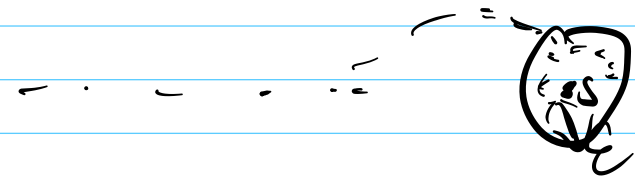
sequence.

③ $\limsup S_n = \sup S$.

④ As the definition of limsup

$$\limsup S_n = \lim_{N\to\infty} \sup\{S_n : n>N\}$$

$a_1, a_2 \dots a_3$



$\{S_n : n>N\}$ is in the circle of $|S_n - s| < \varepsilon$, where s is $\lim S_n$.

$\Rightarrow$ ~~该~~ limsup is the sup of the set of the very tail of the sequence.