HW4.

#12.10

12.10 Prove (s_n) is bounded if and only if $\limsup |s_n| < +\infty$.

Proof: "$\Rightarrow$"

Since (s_n) is bounded,

$\Rightarrow$, $\exists N>0$,

$|s_n| \le M$, $\forall n > N$.

$\exists M>0$

$\Rightarrow$

Then $\forall \varepsilon > 0$ $M \in \mathbb{R}$

$|s_n| \le M + \varepsilon$ $M > 0$

$\Rightarrow$ For any $K > 0$, $\forall n > N$, Let $K = M$

$|s_n| \le M$

As definition

$\Rightarrow$ $\limsup |s_n| \ne +\infty$ $\Rightarrow$ $< +\infty$

"$\Leftarrow$"

$\limsup |s_n| < +\infty$

Let assum $\limsup |s_n| = L$, L

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists N > 0$, for any $n > N$

$\big| |s_n| - L \big| < \varepsilon$

$\Rightarrow$ $|s_n| < L + \varepsilon$

also $0 < |s_n|$

$\Rightarrow$ (s_n) is bounded. proved

$\frac{1}{n}, 1-\frac{1}{n}$

$n\sigma_n = S_1 + S_2 + \dots S_n$

12.12 Let (s_n) be a sequence of nonnegative numbers, and for each n define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.

$\sup\{\sigma_n : n > M\} \le \sup\{\sigma_n : n > N$

(a) Show

$$\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n.$$

$n > N \quad S_1, S_2,$

Hint: For the last inequality, show first that $M > N$ implies

$$\sup\{\sigma_n : n > M\} \le \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}.$$

$\sigma_M \le \frac{N\sigma_N}{M} + \sup\{S_n : n > N$

(b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

(c) Give an example where $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.

12.12

a) Since clearly $\liminf S_n \le \limsup S_n$

and $\liminf \sigma_n \le \limsup \sigma_n$

Just need to show ① $\liminf \sigma_n \ge \liminf s_n$

② $\limsup S_n \ge \limsup \sigma_n$

For ② Let $n > M > N$

$\Rightarrow \sigma_n = \frac{1}{n}(S_1 + S_2 \dots S_N + S_{N+1} + \dots S_n)$

$= \frac{1}{n}(S_1 + S_2 \dots S_N) + \frac{1}{n}(S_{N+1} \dots + S_n)$

Since $\frac{1}{n}(S_{N+1} + \dots S_n) < \frac{1}{n-N}(S_{N+1} + \dots + S_n)$

$< \frac{1}{n-N} \cdot (n-N) \cdot \sup\{S_n ; n > N\}$

$= \sup\{S_n : n > N\}$ ∗

and Since $\frac{1}{n}(S_1 + S_2 \dots + S_N) < \frac{1}{M}(S_1 + S_2 \dots S_N)$ ∗∗

With ∗ and ∗∗

we get $\sup\{\sigma_n : n > M\} \le \frac{1}{M}(S_1 \dots + S_N) + \sup\{S_n : n > N\}$

Then when $N \to +\infty$, $\Rightarrow M \to +\infty$, $\frac{1}{M} \to 0$

$\limsup \sigma_n \le 0 + \limsup S_n$ proved

Then for ① $\liminf \sigma_n \geq \liminf s_n$

Since for every seq, $\{s_n\}$, $\liminf s_n = -\lim\sup(-s_n)$

$$\Rightarrow \begin{cases} \liminf \sigma_n = -\limsup(-\sigma_n) \\ \liminf s_n = -\liminf(-s_n) \end{cases}$$

log in to

Need to show $\limsup(-\sigma_n) \leq \limsup(-s_n)$

$$-\sigma_n = +\frac{1}{n}(-s_1 - s_2 + \dots - s_N) + \frac{1}{n}\cdot(-s_{N+1} - s_{N+2} \dots - s_n)$$

$$\geq \quad + \frac{1}{m}(s_1 + s_2 \dots s_N) + \sup\{-s_n : n \geq N\}$$

b) Since $\lim s_n$ exist.

$\Rightarrow \limsup s_n = \liminf s_n$

$\Rightarrow$ the result from part a) become

$\liminf s_n = \liminf \sigma_n = \limsup \sigma_n = \limsup s_n$

$\Rightarrow \lim \sigma_n$ exist and $\lim \sigma_n = \lim s_n$.

c) Let $s_n = (-1)^n + 1$ Then $\liminf s_n = 0$

$\limsup s_n = 2$

$\Rightarrow \lim s_n$ not exist

Then $\sigma_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 1-\frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$

$\Rightarrow \lim \sigma_n = \limsup = \liminf = 1$

14.2

14.2 Repeat Exercise 14.1 for the following.

(a) $\sum \frac{n-1}{n^2}$ (b) $\sum(-1)^n$

(c) $\sum \frac{3n}{n^3}$ (d) $\sum \frac{n^3}{3^n}$

(e) $\sum \frac{n^2}{n!}$ (f) $\sum \frac{1}{n^n}$

(g) $\sum \frac{n}{2^n}$

a) $\sum \frac{n-1}{n^2}$

Since $\sum \frac{n-1}{n^2} \geq \sum \frac{n-\frac{n}{2}}{n^2} = \sum \frac{\frac{n}{2}}{n^2} = \sum \frac{1}{2n}$

$\because \sum \frac{1}{2n}$ is divergent.

$\Rightarrow \sum \frac{n-1}{n^2}$ is divergent.

b) $\sum (-1)^n$ Since $\lim_{n\to\infty} (-1)^n \neq 0$

$\Rightarrow$ divergent

c) $\sum \frac{3n}{n^3} = \sum \frac{3}{n^2}$ $\int_1^\infty \frac{3}{x^2} = -3x^{-1}\Big|_1^\infty$

$= 0 - (-3) = 3$

$\Rightarrow$ by intergral test

convergent to 3

d) $\sum \frac{n^3}{3^n}$

$$\lim \frac{\left|\frac{(n+1)^3}{3^{n+1}}\right|}{\left|\frac{n^3}{3^n}\right|} = \lim \left|\frac{(n+1)^3}{3(n)^3}\right| = \lim \left|\frac{1}{3}\left(\frac{n+1}{n}\right)^3\right| = \frac{1}{3} < 1$$

$\Rightarrow$ convergent

e) $\sum \frac{n^2}{n!}$

$$\lim \left|\frac{\frac{(n+1)^3}{(n+1)!}}{\frac{n^3}{n!}}\right| = \lim \left|\frac{1}{(n+1)} \cdot \frac{(n+1)^3}{n^3}\right| = 0$$

converges

f) $\sum \frac{1}{n^n} = \sum \left(\frac{1}{n}\right)^n$

$$\lim \left|\left(\frac{1}{n}\right)^n\right|^{\frac{1}{n}} = \lim \frac{1}{n} = 0$$

$\Rightarrow$ convergent

g) $\sum \frac{n}{2^n}$

$$\lim \left|\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right| = \lim \left|\frac{1}{2} \cdot \frac{n+1}{n}\right| = \frac{1}{2}$$

, convergent

14.10

14.10 Find a series $\sum a_n$ which diverges by the Root Test but for which the Ratio Test gives no information. Compare Example 8.

Let $\boxed{\sum_{n=1}^{\infty} 2^{(-1)^n+n} = \sum a_n}$

with root test

$$\lim_{n\to\infty} \left|2^{(-1)^n+n}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\left|2^{(-1)^n\cdot\frac{1}{n}+1}\right| = 2^1 > 1 \text{ , diverge}$$

with Ratio Test

$$\{a_n\} = \cancel{2}, 1, 8, 4, 32,$$

$$\frac{1}{2} < \liminf\left|\frac{a_{n+1}}{a_n}\right| < 1 < \limsup\left|\frac{a_{n+1}}{a_n}\right| < 8.$$

$\Rightarrow$ with Ratio test, give no information

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Rudin
Ch3
6.

a) $a_n = \sqrt{n+1} - \sqrt{n}$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{2\sqrt{n+1}}$$

$\because \sum \frac{1}{2\sqrt{n+1}}$ with integral test

is divergent

$\Rightarrow \sum \sqrt{n+1} - \sqrt{n}$ is divergent

b)

$$\sum \frac{\sqrt{n+1}-\sqrt{n}}{n}$$

$$\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \frac{1}{2n\sqrt{n}} = \frac{1}{2} \cdot n^{-\frac{3}{2}}$$

with integral test for $\sum \frac{1}{2} \cdot n^{-\frac{3}{2}}$

$\sum \frac{1}{2} \cdot n^{-\frac{3}{2}}$ is converget

$\Rightarrow \sum \frac{\sqrt{n+1}-\sqrt{n}}{n} < \sum \frac{1}{2} n^{-\frac{3}{2}}$ is convergent

c) $a_n = (\sqrt[n]{n} - 1)^n$

$$\lim |a_n|^{\frac{1}{n}} = \lim |\sqrt[n]{n} - 1|$$

$$= \lim \sqrt[n]{n} - 1$$

$$= 0$$

$\Rightarrow$ convergent

d) $a_n = \frac{1}{1+z^n}$ for complex values of

ch3 7.

Let $b_n = \frac{\sqrt{a_n}}{n}$

Then ratio test

7. Prove that the convergence of Σa_n implies the convergence of

$$\Sigma \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

$$\lim \left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{\frac{\sqrt{a_{n+1}}}{n+1}}{\frac{\sqrt{a_n}}{n}} \right| = \lim \left| \frac{n}{n+1} \right| \cdot \lim \sqrt{\frac{a_{n+1}}{a_n}}$$

$$= \lim \sqrt{\frac{a_{n+1}}{a_n}}$$

if $\sum a_n$ is convergent.

then

$$\lim \frac{|a_{n+1}|}{|a_n|} < 1 \Rightarrow \lim \sqrt{\frac{|a_{n+1}|}{|a_n|}} < 1$$

$$\Rightarrow \cancel{\Sigma b_n <} \lim \left| \frac{b_{n+1}}{b_n} \right| < 1$$

$\Rightarrow$ proved

ch3 11.

11. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and Σa_n diverges.

(a) Prove that $\Sigma \frac{a_n}{1+a_n}$ diverges.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\Sigma \frac{a_n}{s_n}$ diverges.

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\Sigma \frac{a_n}{s_n^2}$ converges.

(d) What can be said about

$$\Sigma \frac{a_n}{1+na_n} \quad \text{and} \quad \Sigma \frac{a_n}{1+n^2a_n}?$$

a) $\frac{a_n}{1+a_n} = 1 - \frac{1}{1+a_n}$ Since $a_n > 0$.

$\Rightarrow 1 - \frac{1}{1+a_n} > 0$

$\Rightarrow \sum \frac{a_n}{1+a_n}$ diverge

b) right side

$1 - \frac{S_N}{S_{N+k}} = \frac{a_{N+1} + a_{N+2} \dots a_{N+k}}{S_{N+k}}$

$= \frac{a_{N+1}}{S_{N+k}} + \frac{a_{N+2}}{S_{N+k}} \dots \dots \frac{a_{N+k}}{S_{N+k}}$ ①

Since $a_n > 0$ and $S_n = \sum a_n$

$\Rightarrow$ clearly $S_{n+1} > S_n$ for any n

$\Rightarrow S_{N+k} > S_{N+k-1} > \dots > S_{N+2} > S_{N+1}$

$\Rightarrow \frac{a_{N+1}}{S_{N+k}} + \frac{a_{N+2}}{S_{N+k}} \dots \frac{a_{N+k}}{S_{N+k}}$

$\leq \frac{a_{N+1}}{S_{N+1}} + \frac{a_{N+2}}{S_{N+2}} \dots \frac{a_{N+k}}{S_{N+k}}$

"=" get when $k=1$

Then proved.

With this result, when $N=1$, $C_n = \frac{a_n}{S_n}$

left side is $\sum C_n \geq 1-$

right side is