13.3 Let B be the set of all bounded sequences $\boldsymbol{x} = (x_1, x_2, \ldots)$, and define $d(\boldsymbol{x}, \boldsymbol{y}) = \sup\{|x_j - y_j| : j = 1, 2, \ldots\}$.

(a) Show d is a metric for B.

(b) Does $d^*(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{\infty} |x_j - y_j|$ define a metric for B?

(a) $d(x,x) = \sup\{|x_j - x_j| : j = 1, 2, \ldots\}$

$= \sup\{0\} = 0$

$d(x,y) = \sup\{|x_j - y_j|\}$

$= \sup\{|y_j - x_j|\}$

$= d(y,x)$

$d(x,y) + d(y,z) = \sup\{|x_j - y_j|\} +$

$\sup\{|y_j - z_j|\}$

Let $\sup\{|x_j - y_j|\} = \cancel{\sup}|x_k - y_k|$ for some $k, p, m \in \{1, 2, 3 \ldots\}$

and $\sup\{|y_j - z_j|\} = |y_p - z_p|$

$\sup\{|x_j - z_j|\} = |x_m - z_m|$

$\Rightarrow \sup\{|x_j - y_j|\} + \sup\{|y_j - z_j|\}$

$= |x_k - y_k| + |y_p - z_p|$

$\geq |x_m - y_m| + |y_m - z_m| \geq |x_m - z_m|$

$\Rightarrow d(x,y) + d(y,z) \geq d(x,z)$

b) clearly $d^*(x,y)$ satisfy $d(x,y) = d(y,x)$

$d(x,x) = 0$

For $d^*(x,y) + d^*(y,z) \geq d^*(x,z)$

$d^*(x,y) + d^*(y,z) = \sum_{j=1}^{\infty} \left(|x_j - y_j| + |y_j - z_j| \right)$

$\because$ for every term

$|x_j - y_j| + |y_j - z_j| \geq |x_j - z_j|$

$\Rightarrow \sum \left(|x_j - y_j| + |y_j - z_j| \right) \geq \sum |x_j - z_j|$

$\Rightarrow$ proved

13.5 **(a)** Verify one of DeMorgan's Laws for sets:

$$\bigcap\{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcup\{U : U \in \mathcal{U}\}.$$

(b) Show that the intersection of any collection of closed sets is a closed set.

(a) for $x \in \bigcap\{S \setminus U\}$

$x \in S / U_i \quad i \in I$

$\Rightarrow x \notin \bigcup U_i$ for $i \in I$,

$\Rightarrow x \in \{S / \bigcup \{U; U \in \mathcal{U}\}\}$

$\Rightarrow$ proved

(b) With the result of part (a)

let $\{U : U \in \mathcal{U}\}$ be the collection of closed set

From a), we can get

$\bigcap \{ \quad U : U \in \mathcal{U}\}$

$= S \,/\, \bigcup\{S-U : U \in \mathcal{U}\}$

Then hence need to show

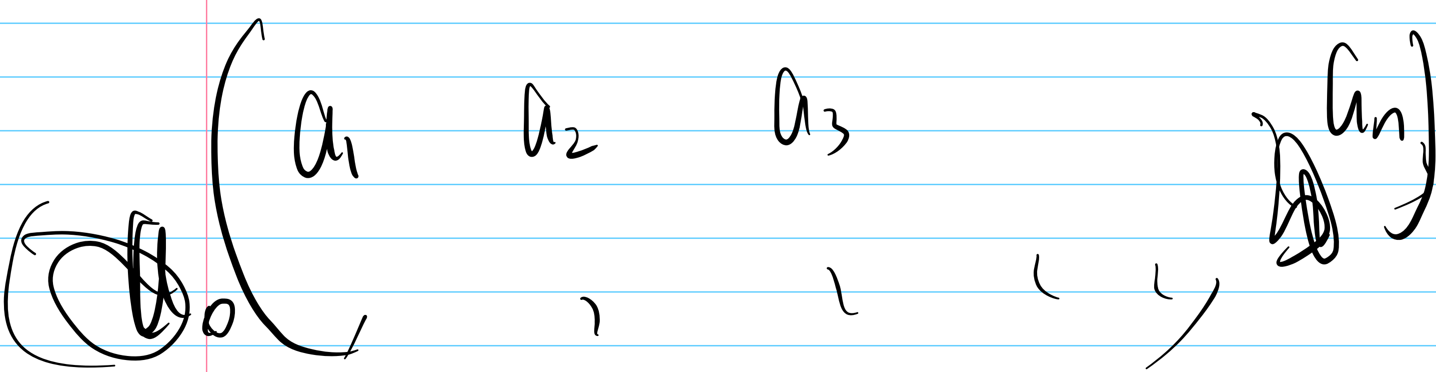
$S - \bigcup\{S-U : U \in \mathcal{U}\}$ is closed

$\because \{U : U \in \mathcal{U}\}$ is the collection of closed set

$\Rightarrow \{S-U : U \in \mathcal{U}\}$ is an open set

$\Rightarrow \bigcup\{S-U,\ U \in \mathcal{U}\}$ is an open set

$\Rightarrow S - \bigcup\{S-\mu : \mu \in \mathcal{U}\}$ ~~the~~ is closed.



13.7 Show that every open set in $\mathbb{R}$ is the disjoint union of a finite or infinite sequence of open intervals.

Need to show for even $S \subseteq R$.

where S is open

$S = \bigcup u_i$ when $\bigcap u_i \bigcap u_2 = \varnothing$

u_i is open interval $\subseteq S$

Since S is open

for every $p \in S$, $\exists B_r(p)$

Since u_i is open interval.

Let $u_i = (a_i, a_{i+1})$, for any u_i

$\Rightarrow \exists B_{r_i}(p) < r$ where $r_i = \frac{a_{i+1} - a_i}{2}$

$p = \frac{a_i + a_{i+1}}{2}$

Clearly for every p, express it as $\frac{a_i + a_{i+1}}{2}$

Let $r_i = \frac{a_{i+1} - a_i}{2}$,

$\Rightarrow u_i$ is the open set Since

$\exists B_{r_i}(p) < r_i$

4. Recall that in class, given (X, d) a metric space, and S a subset of X, we defined the closure of S to be \bar S = \{ p \in X \mid \text{there is a subsequence (p_n)in S that converge to p\}

Prove that taking closure again won't make it any bigger, i.e, if $S_1 = \bar{S}$, and $S_2 = \bar{S}_1$, then $S_1 = S_2$.

For " $S_1 \subseteq S_2$ " $\quad S_1 = \bar{S}$

Let $x \in S_1$

$\exists$ a seq $\{p_n\}$ in S.

s.t $p_n \to x$

$n > N_1 \Rightarrow p_n \in B_{r_1}(x)$

$\because S_2 = \bar{S_1}$

$\Rightarrow \exists$ a seq $\{a_n\}$ in S_1

Let $a_n = x$ for $n = 1, 2, \ldots$

$\Rightarrow a_n \to x$

$\Rightarrow x \in S_2$

$\Rightarrow$ for every $x \in S_1$,

$x \in S_2$

"

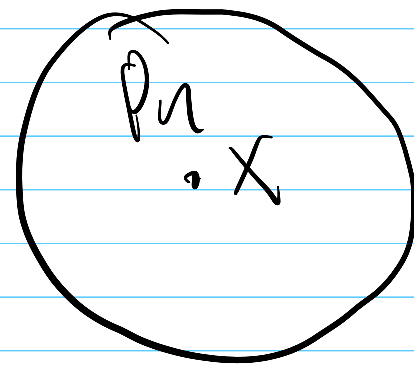
For $S_1 \geq S_2$

Let $x \in S_2$

$\Rightarrow \exists \{p_n\}$ in S_1, s.t.

$p_n \to x$

$\Rightarrow p_n \in B_r(x)$



$\because S_1 = \bar{S}$

$\Rightarrow S_1 = \{m \in X \mid \exists \{a_n\}$ in S, s.t. $a_n \to m\}$

$\because p_n \in S_1 \Rightarrow \exists \{a_n\}$ in S, s.t.

$a_n \to p_n$

$\Rightarrow a_n \in B_{r'}(p_n)$

$\Rightarrow a_n \in B_r(x)$

$\Rightarrow a_n \to x$.

$\Rightarrow x \in S_1$

$\Rightarrow S_1 \supseteq S_2$

Then Above all we get $S_1 = S_2$

5. Prove that $\bar{S}$ is the intersection of all closed subsets in X that contains S. (you may assume result in 4, namely, $\bar{S}$ is closed)

one of DeMorgan's Laws for sets:

$$\bigcap\{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcup\{U : U \in \mathcal{U}\}.$$