Jie Zheng HW6_104

1.

1. In class, we proved that $[0,1]$ is sequentially compact, can you prove that $[0,1]^2$ in $\mathbb{R}^2$ is sequentially compact? (In general, if metric space X and Y are sequentially compact, we can show that $X \times Y$ is sequentially compact.)

Need to prove: If $X \subseteq R$, $Y \subseteq R$ are sequentially compact,

then $X \times Y$ is also sequential compact

$\because$ X, Y is sequentially compact

$\Rightarrow \forall x_n \in X$, $\exists$ $x_{(n_k)}$ and $x' \in X$,

s.t $(x_{(n_k)}) \to x'$

and

$\forall y_n \in Y$, $\exists$ $y_{(m_k)}$ and $y' \in Y$.

s.t $(y_{(m_k)}) \to y'$

Then Let $\{t_n\} = \cancel{(X, Y)}$ $X \times Y = (x_i, y_i)$

we can find subsequence

$(t_n)_p = (x_p, y_p)$ where $p \in n_k \cup m_k$

$\Rightarrow$ s.t $x_p \to x'$, $y_p \to y'$

$\Rightarrow \{t_{n_p}\} \to (x', y')$

$\Rightarrow X \times Y$ is sequential compact.

2\.

2\. Let E be the set of points $x \in [0,1]$ whose decimal expansion consist of only 4 and 7 (e.g. 0.4747744 is allowed), is E countable? is E compact?

Is E countable? uncountable

Pf: Assume $E = \{ x \mid x \in [0,1]$ with decimal expansion consist only $4, 7 \}$

for every digit after "0." there are two choic "4" and "7", and different choice in any digit make a ~~different~~ different point then othere

$\Rightarrow$ for a point with n digit.

there is 2^n ~~pos~~ choices of differen point

Since for $x \in [0,1]$, n can be infinite.

$\Rightarrow 2^n$ is infinite.

$\Rightarrow E$ is uncountable

is E compact.?

- Cleary E is bounded since $x \in [0,1]$
- To check E is closed

it is to check E^c is open

clearly $E^c = \{x \mid x \in [0,1]$ with that not all the digit is ~~bother~~ either 4, or 7.$\}$

for any $x \in E^c$
$\Rightarrow$ Let the first digit not "4", "7" in the k-th posith.
and Let r = the decimal part after k-th poisition.

like.

$x = 0.47732740\ldots$

$r = 0.00002740\ldots$

Then $B_r x = (0.4730000, 0.47354\ldots)$
$\in E^c$

This hold for any $x \in E^c$

$\Rightarrow E^c$ is open

$\Rightarrow E$ is closed

Above. E is compact

3. Let $A_1, A_2, \cdots$ be subset of a metric space. If $B = \cup_i A_i$, then $\bar{B} \supset \cup_i \bar{A}_i$. Is it possible that this inclusion is an strict inclusion?

Let $A_i = \frac{1}{i}$

$\Rightarrow \bigcup_i A_i = B = \{\frac{1}{i} \mid i = 1, 2, 3 \ldots \}$

Since $A_i = \bar{A}_i$

$\Rightarrow \bigcup_i \bar{A}_i = B$

but $\bar{B} = \{0, \frac{1}{i} \mid i = 1, 2, 3 \ldots\}$

which meanse

$\bar{B} = B \cup \{0\}$

$\Rightarrow$ with this example.

we say it is possible for this inclusion to be strict

× (include but not equal)

4. Last time, we showed that any open subset of $\mathbb{R}$ is a countable disjoint union of open intervals. Here is a claim and argument about closed set: {\em every closed subset of $\mathbb{R}$ is a countable union of closed intervals. Because every closed set is the complement of an open set, and adjacent open intervals sandwich a closed interval.} Can you see where the argument is wrong? Can you give an example of a closed set which is not a countable union of closed intervals? (here countable include countably infinite and finite)

just as the Q_2.

$F = \{x \mid x \in [0,1]$ with only 4, 7, in decimal part $\}$

F is proved to be closed

but it is ~~the~~ union of uncountable discrete points insted of countable closed interval.