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1. If X and Y are open cover compact, can you prove that $X \times Y$ is open cover compact? (try to do it directly, without using the equivalence between open cover compact and sequential compact)

proof:

NTS: $\exists$ finite subcover of $X \times Y$: Z.

S.t for any $(x,y) \in X \times Y$.

$(x,y) \in Z$.

$\because X, Y$ are open cover compact.

$(I$, index$)$

$\Rightarrow \exists X \subset \{Z_m \mid m \in I\}$, and $|Z_k| < \infty$ finite.

and $\exists Y \subset \{Z_n \mid n \in I\}$ and $|Z_n| < \infty$ finite

$\Rightarrow$ Construct $Z = Z_m \times Z_n$

$\Rightarrow Z \subset X \times Y$

$\Rightarrow |Z| = |Z_m| \times |Z_n| < \infty$, finite

$\Rightarrow$ We say, $X \times Y$ is open cover compact.

2. Let $f : X \to Y$ be a continuous map between metric spaces. Let $A \subset X$ be a subset. Decide if the followings are true or not. If true, give an argument, if false, give a counter-example.

a - if A is open, then $f(A)$ is open
b - if A is closed, then $f(A)$ is closed.
c - if A is bounded, then $f(A)$ is bounded.
d - if A is compact, then $f(A)$ is compact.
e - if A is connected, then $f(A)$ is connected.

a) False, For any $f(x)=C$, $C \in \mathbb{Z}$

$f(A) = C$, $\because$ $\{C\}$ is not open, $\Rightarrow$ False.

b) False. $f(x) = e^x$, $A = \mathbb{R}$

$f(A) = (0, \infty)$,

c) ~~Let~~ False. Let $f(x) = \ln x$

$A = (0, 1)$, bounded

but $f(A) = (-\infty, 0)$ unbounded.

d) True.

$\because$ A is compact, can find subsequece $\{b_n\}$ of A, s.t $b_n \to b$ (same limit)

$\Rightarrow f(\{b_n\}) \to f(b)$.

$\Rightarrow f(A)$ is compact

e) True.

Assume $f(A)$ is disconnected and let $f(A) = X \cup Y$,

$\Rightarrow A = f^{-1}(X) \cup f^{-1}(Y)$

$\Rightarrow$ A is disconnected, contradiction $\Rightarrow f(A)$ is connected.

3. Prove that, there is not continuous map $f : [0,1] \to \mathbb{R}$, such that f is surjective. (there is a surjective map from $(0,1) \to \mathbb{R}$ though)

Pf: $\because$ $[0,1]$ compact (closed and bonded)

Assume f is surjectie,

$\Rightarrow$ $f([0,1]) = R$ is compact.

but $f([0,1]) = R$ is not compact actually,

$\Rightarrow$ contradition

$\Rightarrow$ $!\exists$ continuous map f like that.