03 | 25 | 2026 HW8_104

1. Let $f_n(x) = \frac{n+\sin x}{2n+\cos n^2 x}$, show that f_n converges uniformly on $\mathbb{R}$.

$\because 1 \geq \cos n^2 x \geq 0$ and $|\sin x| \leq 1$

$\Rightarrow f(x) = \frac{n+\sin x}{2n+\cos n^2 x} \geq \frac{n-1}{2n+1}$

and $f(x) \leq \frac{n+1}{2n-1}$

$\Rightarrow \frac{n-1}{2n+1} \leq f(x) \leq \frac{n+1}{2n-1}$

$\because \frac{n-1}{2n+1} \to \frac{1}{2}$, $\frac{n+1}{2n-1} \to \frac{1}{2}$.

$\Rightarrow f(x) \to \frac{1}{2}$

Need to Find N, s.t for any $n > N$,

$|f(x) - \frac{1}{2}| < \varepsilon$, $\forall \varepsilon > 0$

$\because |f(x) - \frac{1}{2}| \leq \left|\frac{n+1}{2n-1} - \frac{1}{2}\right| = \frac{\frac{3}{2}}{2n-1}$

$\Rightarrow \frac{\frac{3}{2}}{2n-1} < \varepsilon$

$\Rightarrow n > \frac{1}{2} + \frac{3}{4\varepsilon}$

$\Rightarrow \exists N$, s.t $N > \frac{1}{2} + \frac{3}{4\varepsilon}$,

and $|f(x) - \frac{1}{2}| < \varepsilon$

Then: f_n converge uniformly on $\mathbb{R}$.

2. Let $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Show that the series is continuous on $[-1, 1]$ if $\sum_n |a_n| < \infty$. Prove that $\sum_{n=1}^{\infty} n^{-2} x^n$ is continuous on $[-1, 1]$.

(In general, if one only know that $\sum_n a_n$ and $\sum_n (-1)^n a_n$ converge, then the result still holds, but is harder to prove. See Ross Thm 26.6)

pf: ① With $f(x) = \sum_{n=1}^{\infty} a_n x^n$, and $\sum_n |a_n| < \infty$

NTS: $f(x)$ is continuous on $[-1,1]$

With weierstrass test, $\because \sum_n |a_n| < \infty$,

Then let $M_n = |a_n|$.

For $x \in [-1,1]$, $-1 \le x^n \le 1$

$\therefore |a_n x^n| \le |a_n| = M_n$

$\Rightarrow -a_n \le a_n x^n \le a_n$

$\Rightarrow$ $f(x)$ is uniformly convergent on $[-1,1]$

$\Rightarrow$ $f(x)$ is continous on $[-1,1]$

② $g(x) = \sum_{1}^{\infty} n^{-2} x^n$, $b_n = \sum_n n^{-2}$

consider $b_n = \sum \frac{1}{n^2}$

with integral test b_n is convergent

also for $|b_n| = |\frac{1}{n^2}| = \frac{1}{n^2}$ is convergent

$\Rightarrow b_n$ is absolutly converergent.

with the result from above, $g(x)$ is continuous on $[-1,1]$

3. Show that $f(x) = \sum_n x^n$ represent a continuous function on $(-1,1)$, but the convergence is not uniform. (Hint: to show that $f(x)$ on $(-1,1)$ is continuous, you only need to show that for any $0 < a < 1$, we have uniform convergence on $[-a,a]$. Use Weierstrass M-test.)

For any $a \in [0,1)$, Need to show $f(x)$ is continuous on $[-a,a]$

$|f_n(x)| = |x^n| = |x|^n$

$\because$ $x \in [-a,a]$

$\Rightarrow$ $|f_n(x)| \leq a^n$

$\therefore |f(x)| = \sum |f_n(x)| \leq \sum a^n = a^0 + a^1 + \dots a^n = \frac{1-a^n}{1-a}$

~~Fact~~ $\because a \in [0,1)$

$\therefore \frac{1-a^n}{1-a} \to \frac{1}{1-a}$.

$\therefore |f(x)| \leq \frac{1}{1-a}$ $\Rightarrow$ uniform convergence on $[-a,a]$

$\Rightarrow$ $f(x)$ is continuous on $[-a,a]$

Then Need to show it is not uniform

Then it is to find an $\varepsilon > 0$, for any N

s.t $|f_n(x) - f(x)| > \varepsilon$. for $n > N$.

$\Rightarrow \left| \frac{1-x^n}{1-x} - \frac{1}{1-x} \right| > \varepsilon$, $\because x \in (-1,1)$

$\Rightarrow \frac{x^n}{1-x} > \varepsilon$

IDK how to continue.

$x^n > (1-x)\varepsilon$, as $x \in (0,1)$

$n < \log_x (1-x)\varepsilon$