- Read Ross p257, Example 3 about smooth interpolation between 0 for $x \le 0$ and $e^{-1/x}$ for $x > 0$. Construct a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $x \le 0$ and $f(x) = 1$ for $x \ge 1$, and $f(x) \in [0,1]$ when $x \in (0,1)$.

From exaple 3.

$$g(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

is smooth for any order of derivatio

$\Rightarrow g^{(n)}(x)$ exist for any n

consider $g(1-x)$

$$t(x) = g(1-x) = \begin{cases} e^{-\frac{1}{1-x}} & x < 1 \\ 0 & x \ge 1 \end{cases}$$

$\because$ $g(x)$ is smooth at R, $\Rightarrow$ $t(x)$ should be smooth at R

Let $f(x) = \dfrac{g(x)}{t(x) + g(x)}$

① for $x \ge 1$,

$f(x) = \dfrac{g(x)}{g(x)} = 1$

② for $x \le 0$

$f(x) = \dfrac{0}{t(x)} = 0$

③ for $x \in (0,1)$

$f(x) = \dfrac{e^{-\frac{1}{x}}}{e^{-\frac{1}{x}} + e^{-\frac{1}{1-x}}}$

clearly $f(x) > 0$,

and $f(x) < \dfrac{e^{-\frac{1}{x}}}{e^{-\frac{1}{x}}} = 1$

$\Rightarrow f(x) \in (0,1)$

- Rudin Ch 5, Ex 4 (hint: apply Rolle mean value theorem to the primitive)

4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Consider $f(x) = C_0 x + \frac{C_1}{2} x^2 + \frac{C_2}{3} x^3 \ldots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n+1} x^{n+1}$

clearly $f(x)$ is differential at $[0,1]$

and $f(0)=0$, $f(1) = C_0 + \frac{C_1}{2} + \ldots \frac{C_{n-1}}{n} + \frac{C_n}{n+1}$
$= 0$

$\Rightarrow f(0) = f(1)$.

With Rolle theorem

$\exists p \in [0,1]$ such that

$f'(p) = 0$

$\Rightarrow f'(x) = C_0 + C_1 x \ldots C_{n-1} x^{n-1} + C_n x^n$

For $\exists p \in [0,1]$, s.t

$f'(p) = 0$

Then proved

8. Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \varepsilon \qquad (1)$$

whenever $0 < |t-x| < \delta$, $a \le x \le b$, $a \le t \le b$. (This could be expressed by saying that f is *uniformly differentiable* on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

- Rudin Ch 5, Ex 8 (ignore the part about vector valued function. Hint, use mean value theorem to replace the difference quotient by a differential)

Wlog, let $a \le x < t \le b$

$\Rightarrow$ with mean value theorem.

$\exists p \in [x, t]$, s.t

$$f'(p) = \frac{f(t)-f(x)}{t-x}$$

$$\Rightarrow \left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| = \left|f'(p) - f'(x)\right| \qquad (2)$$

$\because f'$ is continuous, $\Rightarrow f'(x)$ is uniformly continuous

$\Rightarrow |f'(m) - f'(n)| < \varepsilon$, $\exists \delta$, s.t $|m-n| < \delta$

$\forall \varepsilon > 0$,

Then for that δ

$|t-x|<\delta$, $\Rightarrow |p-x|<\delta$

$\Rightarrow$ we get

$2 < \varepsilon$

$\Rightarrow$ proved.

- Rudin Ch 5, Ex 18 (alternative form for Taylor theorem)

18. Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α, β, and P be as in Taylor's theorem (5.15). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

From Talor theorem

$$P_\alpha(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k$$

$$= f(\alpha) + \sum_{k=1}^{n-1} \frac{f^{k}(\alpha)}{k!}(\beta-\alpha)^k \quad (1)$$

Differentiate $f(t) - f(\beta) = (t-\beta)Q(t)$ for n times

Left side $= f^{(n)}(t)$

right :

1st derivative

$Q(t) + (t-\beta)Q^{(1)}(t)$

2st derivative

$Q^{(1)}(t) + Q^{(1)}(t) + (t-\beta)Q^{t}(\beta)$

$= 2Q^{(1)}(t) + (t-\beta)Q_c^{(2)}(t)$

Guess k-time derivative

$$= kQ^{(k-1)}(t) + (t-\beta)Q^{(k)}(t)$$

To check this:

Base Case, for $k=0$ and $k=1$.

already proved above

~~A~~ssume for $n = k-1$ hold

$$\Rightarrow [(t-\beta)Q(t)]^{[k-1]} = (k-1)Q^{(k-2)}(t) + (t-\beta)Q^{(k-1)}(t)$$

for $n=k$

Dirivative that

can get

$$[(t-\beta)Q(t)]^{(k)} = kQ^{(k-1)}(t) + (t-\beta)Q^{(k)}(t)$$

Then finish the check

$$\Rightarrow f^{(k)}(t) = kQ^{(k-1)}(t) + (t-\beta)Q^{(k)}(t)$$

With (1)

$$R(\beta) = f(\alpha) + \sum_{k=1}^{n-1} \frac{k\,Q^{(k-1)}(\alpha) + (\alpha-\beta)\,Q^{k}(\alpha)}{k!} \cdot (\beta-\alpha)^{k}$$

$$= f(\alpha) + \sum_{k=1}^{n-1} \frac{Q^{(k-1)}(\alpha)}{(k-1)!}(\beta-\alpha)^{k} - \sum_{k=1}^{n-1} \frac{Q^{k}(\alpha)}{k!}(\beta-\alpha)^{k+1}$$

$$= f(\alpha) + \sum_{k=0}^{n-2} \frac{Q^{(k)}(\alpha)}{k!}(\beta-\alpha)^{k+1} - \sum_{k=1}^{n-1} \frac{Q^{k}(\alpha)}{k!}(\beta-\alpha)^{k+1}$$

$$= f(\alpha) + \frac{Q^{(0)}(\alpha)}{0!}(\beta-\alpha)^{0+1} - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n}$$

$$= f(\alpha) + \frac{f(\alpha)-f(\beta)}{\alpha-\beta} \cdot (\beta-\alpha) - \cdots\cdots$$

$$= f(\beta) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n}$$

Then:

$$f(\beta) = R_\beta + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n}$$

Proved

- Rudin Ch 5, Ex 22

a) assume there are two fixed point, s.t
$f(x) = x$.
$\Rightarrow f(x_1) = x_1$, $f(x_2) = x_2$, Let $x_1 < x_2$
with Mean value theorem
$\exists\, c \in [x_1, x_2]$ s.t
$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = 1$$
contradition with $f'(t) \neq 1$ for every t,
$\Rightarrow$ proved

b) Assume there is a fixed point c.
$\Rightarrow c + c(1+e^t)^{-1} = c$
$\Rightarrow (1+e^t)^{-1} = 0$
Since $(1+e^t)^{-1}$ can not be 0.
$\Rightarrow$ There is no fixed point

c)

$\therefore |f'(t)| \leq A$

with mean value theorem

for any a $(-\infty, \infty)$

$$\left| \frac{f(x) - f(a)}{x - a} \right| \leq A$$