HW10

Jiayin Lin

1 Question 1

$$
\begin{aligned}
\text{(a)} &= \lim_{x\to 0} \frac{\int_0^x e^{t^2}dt - \int_0^0 e^{t^2}dt}{x-0} \\
&= \frac{d}{dx}\int_0^x e^{t^2}dt|_{x=0} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\text{(b)} &= \frac{d}{dx}\int_0^x e^{t^2}dt|_{x=3} \\
&= e^9
\end{aligned}
$$

2 Question 2

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x+1} f(t)dt - \int_{-\infty}^{x-1} f(t)dt \\
F'(x) &= f(x+1) - f(x-1)
\end{aligned}
$$

3 Question 3

$$
\begin{aligned}
\int_0^1 x\sqrt{1-x^2}dx &= -\frac{1}{2}\int_1^0 \sqrt{1-x^2}d1-x^2 \\
&= \frac{1}{3}
\end{aligned}
$$

4 optional 1

$$\begin{aligned}\int_a^b f'(x)(xf(x))dx &= 0 - \int_a^b f(x)(f(x)+xf'(x))dx \\ &= -\frac{1}{2}\int_a^b f(x)^2 dx \\ &= -\frac{1}{2}\end{aligned}$$

cauchy-schwartz with vector space as $C^1[a,b]$

$$\int_a^b f'(x)^2 dx \int_a^b (xf(x))^2 dx > (\int_a^b f'(x)(xf(x))dx)^2 = \frac{1}{4}$$

5 optional 2

$$\begin{aligned}s\int_1^\infty \frac{[x]}{x^{s+1}}dx &= \sum_{k=1}^\infty s\int_k^{k+1} \frac{k}{x^{s+1}}dx \\ &= \sum_{k=1}^\infty k((\frac{1}{k})^s - (\frac{1}{k+1})^s) \\ &= \sum_{k=1}^\infty k(\frac{1}{k})^s - \sum_{k=1}^\infty (k-1)(\frac{1}{k})^s \\ &= \sum_{k=1}^\infty \frac{1}{k^s} = \zeta(s)\end{aligned}$$

$$\begin{aligned}s\int_1^\infty \frac{[x]}{x^{s+1}}dx &= s\int_1^\infty \frac{x}{x^{s+1}}dx - s(\int_1^\infty \frac{x}{x^{s+1}}dx - \int_1^\infty \frac{[x]}{x^{s+1}}dx) \\ &= \int_1^\infty \frac{s}{x^s}dx - s(\int_1^\infty \frac{x-[x]}{x^{s+1}}dx) \\ &= \frac{s}{s-1} - s(\int_1^\infty \frac{x-[x]}{x^{s+1}}dx)\end{aligned}$$

6 optional 3

Since α is obviously monotone, we just need α continuous at 0 because then we can split the integral into two parts [0,x] and (x,1] and let x be enough small.
Take any $\varepsilon > 0$, then we can take $N = [\log_{0.5}\varepsilon] + 1$ so we have $2^{-N} < \varepsilon$. Then the delta is just $\frac{1}{N}$ to finish the proof.