Quiz 1

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1 Question 1

consider

$$f(x) = e^{\frac{1}{x(x-1)}} \text{ for } x \in (0,1)$$

we have

$$f'(x) = f(x)\frac{(2x-1)}{(x-x^2)^2}$$

and

$$f'(0) = \lim_{x\to 0} \frac{f(x)-0}{x} = 0 \text{ and same for } f'(1)$$

induction by supoose

$$f^{(n)}(x) = f(x)Q_n(x) \text{ for some rational function } Q_n(x)$$

and

$$f^{(n)}(0) = 0$$
$$f^{(n)}(1) = 0$$

find the n+1 derivative

$$\begin{aligned} f^{(n+1)}(x) &= f'(x)Q_n(x) + f(x)Q_n'(x) \\ &= f(x)Q_1(x)Q_n(x) + f(x)Q_n'(x) \\ &= f(x)Q_{(n+1)}(x) \end{aligned}$$

for

$$Q_{(n+1)}(x) = Q_1(x)Q_n(x) + Q_n\prime(x)$$

finish the first induction, and now look at endpoints

$$\begin{aligned} f^{(n+1)}(0) &= \lim_{x\to 0}(f^{(n)}(x) - f^{(n)}(0))\frac{1}{x} \\ &= \lim_{x\to 0^+}(f^{(n)}(x) - f^{(n)}(0))\frac{1}{x} \\ &= \lim_{x\to 0^+} e^{-\frac{1}{x(x-1)}}\frac{Q_n(x)}{x} \\ &= 0 \text{ by L'hopital rule} \end{aligned}$$

the same reason for $f^{(n+1)}(1)$

Now we let this function to be g

$$g(x) = \frac{\int_0^x f(t)dt}{\int_0^1 f(t)dt} \text{ for } x \in (0,1) \text{ , 0 for } x \leq 0 \text{ , 1 for } x \geq 1$$

This integral obviously will satisfy the requirement by Newton-Lebeiz rule

2 Question 2

$$\text{let } f(x) = c_0 x + \frac{c_1}{2}x^2 + \cdots + \frac{c_n}{n+1}x^{n+1}$$

then f is differentiable and

$$\begin{aligned} f(0) &= 0 \\ f(1) &= c_0 + \frac{c_1}{2} + \cdots + \frac{c_n}{n+1} = 0 \\ f'(x) &= c_0 + c_1 x + \cdots + c_n x^n \end{aligned}$$

so the equation has a real root in [0,1]

3 Question 3

by Lagrange Mean Value Theorem

$$|\frac{f(t)-f(x)}{t-x} - f'(x)| = |f'(u) - f'(x)| \text{ for } u \in [x,t]$$

hence

$$|u - x| < \delta$$

by continuity it is less than ε

4 Question 4

$$f(t) - f(\beta) = Q(t)(t - \beta)$$

differentiate n times

$$f^{(n)}(t) = Q^{(n)}(t)(t - \beta) + (n)Q^{(n-1)}(t)$$

induction on a obvious base case n=1

$$= Q^{(n)}(t)(t - \beta) + \frac{n!(f(\beta) - P(\beta))}{(\beta - t)^n}$$

rearrange

$$f^{(n)}(t)\frac{(\beta - t)^n}{n!} = Q^{(n)}(t)(t - \beta)\frac{(\beta - t)^n}{n!} + f(\beta) - P(\beta)$$

take $t = \alpha$

$$f(\beta) = P(\beta) + f^{(n)}(t)\frac{(\beta - t)^n}{n!} + Q^{(n)}(t)\frac{(\beta - t)^{n+1}}{n!}$$

finish

5 Question 5

5.1 a

if there are 2 fix points a¡b then $\exists c \in [a, b]$ such that

$$\frac{f(a) - f(b)}{a - b} = f'(c) = 1$$

which is contradiction

5.2 b

$$f'(t) = 1 - \frac{e^t}{(1+e^t)^2} < 1$$

but

$$f(t) - t = \frac{1}{(1+e^t)} < 0$$

5.3 c

if not by intermediate value theorem $f(x) > x$ with no loss of generality, but also

$$f(x) - f(0) \leq Ax\text{for x¿0}$$

otherwise contradict $|f'(x)| \leq A$

then we have

$$Ax + f(0) > x$$

take $x = \frac{f(0)}{1-A} > 0$, since A¡1, $f(0) > 0$ we have $f(0) > f(0)$ contradiction

and also

$$x_{n+1} - x = f(x_n) - f(x) \leq (x_n - x)A$$

so the sequence is boudned by a geometric sequence that converge to 0