1.10 $n=1 \Rightarrow \quad 3=3$ ✓ Assume true for $n\le k$

$n=k+1 \Rightarrow (2(k+1)+1)+(2(k+1)+3)+(2(k+1)+5)+\dots+(4(k+1)-1)$

$= (2k+3)+(2k+5)+\dots(4k+3) = 3k^2+(4k+3)-(2k+1)+(4k+1)$

$= 3k^2+6k+3 = 3(k+1)^2$

1.12a. $n=1 \quad a+b = \binom{1}{0}a+\binom{1}{1}b = a+b$

$n=2 \quad (a+b)^2 = a^2+2ab+b^2 = \binom{2}{0}a^2+\binom{2}{1}ab+\binom{2}{2}b^2$

$n=3 \quad (a+b)^3 = a^3+3a^2b+3ab^2+b^3 = \binom{3}{0}a^3+\binom{3}{1}a^2b+\binom{3}{2}ab^2+\binom{3}{3}b^3$

b. $\binom{n}{k}+\binom{n}{k-1} = \frac{n!}{k!(n-k)!}+\frac{n!}{(k-1)!(n-k+1)!} = \frac{n!(n-k+1)+n!k}{k!(n-k+1)!} =$

$\frac{n!(n+1)}{k!(n-k+1)!} = \binom{n+1}{k}$

c. $(a+b)^{n+1} = (a+b)(a+b)^n \overset{\text{using ind. hyp.}}{=} a\left[\binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\dots+\binom{n}{n-1}ab^{n-1}+\binom{n}{n}b^n\right]$

$+b\left[\binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\dots+\binom{n}{n-1}ab^{n-1}+\binom{n}{n}b^n\right] =$

$\binom{n}{0}a^{n+1}+\underbrace{\left(\binom{n}{1}+\binom{n}{0}\right)}_{\text{by part b}}a^nb+\left(\binom{n}{2}+\binom{n}{1}\right)a^{n-1}b^2+\dots+\left(\binom{n}{n}+\binom{n}{n-1}\right)ab^n+\binom{n}{n}b^{n+1}$

$= a^{n+1}+\binom{n+1}{1}a^nb+\dots+\binom{n+1}{n}ab^n+b^{n+1}$

2.1 using Rational Zeros Thm

$x^2-3=0$ candidates $= \{\pm1,\pm3\}$

$x^2-5=0$ " $= \{\pm1,\pm5\}$

$x^2-7=0$ " $= \{\pm1,\pm7\}$

$x^2-24=0$ " $= \{\pm1,\pm2,\pm3,\pm4,\pm6,\pm8,\pm12,\pm24\}$

$x^2-31=0$ " $= \{\pm1,\pm31\}$

no candidates work

2.2 $x^3-2=0$ using Rational Zeros Thm candidates: $\{\pm1, \pm2\}$

$x^2-5=0$ " : $\{\pm1, \pm5\}$ no candidates work

$x^4-13=0$ " : $\{\pm1, \pm13\}$

2.7a $x=\sqrt{4+2\sqrt3}-\sqrt3$

$x+\sqrt3=\sqrt{4+2\sqrt3}$

$x^2+2\sqrt3x+3=4+2\sqrt3$

$x^2+2\sqrt3\,x-1-2\sqrt3=0$

$(x+(1+2\sqrt3))(x-1)=0$

$x=1$

b. $x=\sqrt{6+4\sqrt2}-\sqrt2$

$x+\sqrt2=\sqrt{6+4\sqrt2}$

$x^2+2\sqrt2x+2=6+4\sqrt2$

$x^2+2\sqrt2x-4-4\sqrt2=0$

$(x-2)(x+2+2\sqrt2)=0$

$x=2$

3.1i. $a+c=b+c$

$(a+c)-c=(b+c)-c$

$a+(c-c)=b+(c-c) \Rightarrow a+0=b+0$

$a=b$

ii. $a\cdot0=a\cdot(0+0)=a\cdot0+a\cdot0$

$0=a\cdot0$ by (i)

iii. $(ab+(-a)b=(a+(-a))b=0\cdot b=0=ab+(-(ab))$

$(-a)b=-ab$ from (i)

iv. $(-a)(-b)=-a(-b)$

$=-b(-a)$

$(-a)(-b)+(-a)(b)=(-a)(-b+b)=(-a)\cdot0$

$\Rightarrow 0=(-ab)+ab=\underbrace{(-a)b+ab}_{\text{by iii}} \Rightarrow (-a)(-b)=ab$ by i

v. $ac=bc \Rightarrow ac-bc=\underset{\text{by i}}{bc-bc} \Rightarrow c(a-b)=0 \Rightarrow c\cdot c^{-1}(a-b)=0\cdot c^{-1}$

$\Rightarrow \underset{\text{by ii}}{1\cdot(a-b)=0} \Rightarrow a-b=0 \Rightarrow a-b+b=b \Rightarrow a=b$

vi. $ab=0 \Rightarrow$ WLOG if $b\neq0$, $a\cdot b\cdot b^{-1}=0\cdot b^{-1} \Rightarrow a=0$ WLOG

3.2.i. $a\leq b \Rightarrow a+((-a)+(-b))\leq b+((-a)+(-b))$

$\Rightarrow -b\leq -a$

3.6 a $|a+b+c| = |a+(b+c)| \leq |a| + |b+c|$ by $\Delta$-inequality

$\Rightarrow |a| + |b+c| \leq |a| + |b| + |c|$ by $\Delta$-inequality

b. $n=1$ : $|a_1| \leq |a_1|$

Assume $|a_1 + \dots + a_n| \leq |a_1| + \dots + |a_n|$ for $n \leq k$

Prove for $k+1$: $|a_1 + \dots + a_n + a_{n+1}| \leq |a_1 + \dots + a_n| + |a_{n+1}|$ by $\Delta$-inequality

$\leq |a_1| + \dots + |a_n| + |a_{n+1}|$ by ind. hyp.

4.11 by denseness of $\mathbb{Q}$, $\exists r \in \mathbb{Q}$ s.t. $a < r < b$

Pf by contradiction: assume there are finite number of rationals between a and b

$\Rightarrow \exists r \in \mathbb{Q}$ s.t. $r \geq s$ $\forall s \in \mathbb{Q}, a < s < b$ and $a < r < b$

by denseness of $\mathbb{Q}$, $\exists t \in \mathbb{Q}$ s.t. $r < t < b$ $\Rightarrow$ contradiction

$\therefore$ there are infinitely many rationals between a and b

4.14a Let $b \in B$. $\sup(A+B) \geq a+b$ $\forall a \in A, b \in B$

$\Rightarrow \sup(A+B) - b \geq a$ $\forall a \in A, b \in B$

$\Rightarrow \sup(A+B) - b$ is upper bound for $A$ $\forall b \in B$

$\Rightarrow \sup A \leq \sup(A+B) - b$ $\forall b \in B$

$\Rightarrow \sup(A+B) - \sup A \geq b$ $\forall b \in B$

$\Rightarrow \sup(A+B) - \sup A$ is upper bound for $B$

$\Rightarrow \sup(A+B) - \sup A \geq \sup B$

$\Rightarrow \sup(A+B) \geq \sup A + \sup B$

WTS: $\sup(A+B) \leq \sup A + \sup B$ $\Rightarrow$ WTS $\sup A + \sup B$ is upper bound for $A+B$

$\sup A \geq a$ $\forall a \in A$

$\sup A + b \geq a + b$ $\forall a \in A, b \in B$ $\qquad \sup B \geq b$ $\forall b \in B$

$\sup A + \sup B \geq \sup A + b \geq a+b$ $\forall a \in A, b \in B$

$\Rightarrow \sup A + \sup B$ is upper bound for $A+B$

$\Rightarrow \sup(A+B) \leq \sup A + \sup B$

$\therefore \sup(A+B) = \sup A + \sup B$

b. $\inf(A+B) = -\sup(-(A+B))$ by proof of 4.5, where $-(A+B) = \{-c : c \in A+B\}$

$= -$

$\inf(A+B) \leq a+b \quad \forall a \in A, b \in B$

$\Rightarrow \inf(A+B) - b \leq a \quad \forall a \in A, b \in B$

$\Rightarrow \inf(A+B) - b$ is lower bound for $A$ $\forall b \in B$

$\Rightarrow \inf A \geq \inf(A+B) - b \quad \forall b \in B$

$\Rightarrow \inf(A+B) - \inf A \leq b \quad \forall b \in B$

$\Rightarrow \inf(A+B) - \inf A$ is lower bound for $B$

$\Rightarrow \inf(A+B) - \inf A \leq \inf B$

$\Rightarrow \inf(A+B) \leq \inf A + \inf B$

$\inf A \leq a \ \forall a \in A \qquad \inf B \leq b \ \forall b \in B$

$\inf A + b \leq a+b \quad \forall a \in A, b \in B$

$\inf A + \inf B \leq \inf A + b \leq a+b \quad \forall a \in A, b \in B \Rightarrow \inf A + \inf B$ is lower bound for $A+B$

$\Rightarrow \inf A + \inf B \leq \inf A+B$

$\therefore \inf(A+B) = \inf A + \inf B$

7.5a $\lim_{n\to\infty} \sqrt{n^2+1} - n = \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}} = \frac{n^2+1-n^2}{\sqrt{n^2+1}} = \frac{1}{\sqrt{n^2+1}} \rightarrow 0$

b. $\sqrt{n^2+n} - n = \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n} = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n} \Rightarrow \frac{1}{\frac{1}{n}\sqrt{n^2+n}+1} =$

$\frac{1}{\sqrt{1+\frac{1}{n}}+1} \Rightarrow \frac{1}{2}$

c. $\sqrt{4n^2+n} - 2n = \frac{(\sqrt{4n^2+n}-2n)(\sqrt{4n^2+n}+2n)}{\sqrt{4n^2+n}+2n} = \frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n} = \frac{n}{\sqrt{4n^2+n}+2n}\frac{(\frac{1}{n})}{(\frac{1}{n})}$

$= \frac{1}{\sqrt{4+\frac{1}{n}}+2} \rightarrow \frac{1}{4}$