Math 104 HW 10

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1 33.4

For interval $[a,b]$, let $f(x) = 1$ for rational $x \in [a,b]$ and let $f(x) = -1$ for irrational $x \in [a,b]$. Then $|f|(x) = 1, \forall x \in [a,b]$, so $|f|$ is a constant function and is thus integrable. However, for any partition P, $U(f,P) = b-a$ and $L(f,P) = -(b-a)$, so $U(f) - L(f) = 2(b-a) \neq 0$. Thus, f is not integrable.

2 33.7

2.1 a

$U(f^2,P) - L(f^2,P) = \sum_{k=1}^{n} \sup\{f(x)^2 : x \in [t_{k-1}, t_k]\}(t_k - t_{k-1}) - \inf\{f(x)^2 : x \in [t_{k-1}, t_k]\}(t_k - t_{k-1})$

Let $y_k = x$ s.t. $f(x)^2 = \sup\{f(x)^2 : x \in [t_{k-1}, t_k]\}$, $z_k = x$ s.t. $f(x)^2 = \inf\{f(x)^2 : x \in [t_{k-1}, t_k]\}$

$\sum_{k=1}^{n}(\sup\{f(x)^2 : x \in [t_{k-1}, t_k]\} - \inf\{f(x)^2 : x \in [t_{k-1}, t_k]\})(t_k - t_{k-1}) = \sum_{k=1}^{n}(y_k^2 - z_k^2)(t_k - t_{k-1}) = \sum_{k=1}^{n}(y_k + z_k)(y_k - z_k)(t_k - t_{k-1})$

y_k either equals $\sup\{f(x) : x \in [t_{k-1}, t_k]\}$ or $\inf\{f(x) : x \in [t_{k-1}, t_k]\}$. If $y_k = \sup\{f(x) : x \in [t_{k-1}, t_k]\}$, $z_k \geq \inf\{f(x) : x \in [t_{k-1}, t_k]\}$. If $y_k = \inf\{f(x) : x \in [t_{k-1}, t_k]\}$, $z_k \leq \sup\{f(x) : x \in [t_{k-1}, t_k]\}$. Thus, $y_k - z_k \leq \sup\{f(x) : x \in [t_{k-1}, t_k]\} - \inf\{f(x) : x \in [t_{k-1}, t_k]\}$

$\sum_{k=1}^{n}(y_k + z_k)(y_k - z_k)(t_k - t_{k-1}) \leq \sum_{k=1}^{n}(B+B)(\sup\{f(x) : x \in [t_{k-1}, t_k]\} - \inf\{f(x) : x \in [t_{k-1}, t_k]\})(t_k - t_{k-1}) \leq 2B[U(f,P) - L(f,P)]$

2.2 b

Because f is integrable, $\exists$ partitions P, Q s.t. $U(f,P) - L(f,Q) = 0$
$U(f, P \cup Q) - L(f, P \cup Q) \leq U(f,P) - L(f,Q) = 0$
Thus, $U(f, P \cup Q) - L(f, P \cup Q) = 0$
Thus, from part a, $U(f^2, P \cup Q) - L(f^2, P \cup Q) \leq 2B[U(f, P \cup Q) - L(f, P \cup Q)] =$

0
Thus, $U(f^2) = L(f^2)$

3 33.13

Because f and g are continuous, $h = f - g$ is continuous. By Theorem 33.3, $\int_a^b f - \int_a^b g = \int_a^b (f-g) = \int_a^b h = 0$.

WTS: $\exists x$ s.t. $h(x) = 0$

$\exists P$ s.t. $U(h,P) = \sum_{k=1}^n \sup\{h(x) : x \in [t_{k-1}, t_k]\}(t_k - t_{k-1}) = U(h) = 0$

If $\exists k$ s.t. $\sup\{h(x) : x \in [t_{k-1}, t_k]\} = 0$ then $\exists x$ s.t. $h(x) = 0$
Else, pick x s.t. $h(x) > 0$, y s.t. $h(y) < 0$
Because h is continuous, $[a,b]$ is connected, and $h(y) < 0 < h(x)$, $\exists z \in (a,b)$ s.t. $h(z) = 0$

4 35.4

4.1 a

$\int_0^{\frac{\pi}{2}} x dF(x) = x \sin x|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x = \frac{\pi}{2} + \cos x|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$

4.2 b

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x dF(x) = x \sin x|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \cos x|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2} - \frac{\pi}{2} = 0$

5 35.9a

Let M and m be the max and min values of $f(x)$ on $[a,b]$
$\int_a^b f dF = \sum_{k=0}^n f(t_k)[F(t_k^-) - F(t_k^+)] + \sum_{k=1}^n M(f, (t_{k-1}, t))[F(t_k^-) - F(t_{k-1}^+)] \leq \sum_{k=0}^n M[F(t_k^-) - F(t_k^+)] + \sum_{k=1}^n M[F(t_k^-) - F(t_{k-1}^+)] = M(F(b) - F(a))$
$\int_a^b f dF = \sum_{k=0}^n f(t_k)[F(t_k^-) - F(t_k^+)] + \sum_{k=1}^n m(f, (t_{k-1}, t))[F(t_k^-) - F(t_{k-1}^+)] \geq \sum_{k=0}^n m[F(t_k^-) - F(t_k^+)] + \sum_{k=1}^n m[F(t_k^-) - F(t_{k-1}^+)] = m(F(b) - F(a))$
$m(F(b) - F(a)) \leq \int_a^b f dF \leq M(F(b) - F(a))$
$m \leq \frac{\int_a^b f dF}{F(b) - F(a)} \leq M$
Because f is continuous, by the Intermediate Value Theorem, $\exists x \in [a,b]$ s.t. $f(x) = \frac{\int_a^b f dF}{F(b)-F(a)} \Rightarrow \int_a^b f dF = f(x)[F(b) - F(a)]$