Math 104 HW 1

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1 9.9

1.1 a

$\lim s_n = +\infty \Rightarrow \forall M > 0, \exists N_1\, s.t. \forall n > N_1, s_n > M$
Let $N = max\{N_0, N_1\}$
$\forall n > N, t_n \geq s_n > M \Rightarrow \lim t_n = +\infty$

1.2 b

$\lim t_n = -\infty \Rightarrow \forall M < 0, \exists N_1\, s.t. \forall n > N_1, t_n < M$
Let $N = max\{N_0, N_1\}$
$\forall n > N, s_n \geq t_n < M \Rightarrow \lim s_n = -\infty$

1.3 c

Let $\lim s_n = s, \lim t_n = t$
$\lim(t_n - s_n) = t - s \Rightarrow \forall \epsilon > 0, \exists N_1\, s.t. \forall n > N_1, |(t_n - s_n) - (t - s)| < \epsilon$
For the sake of contradiction, assume $s > t$
Let $N = max\{N_0, N_1\}$
$\forall n > N, t_n - s_n \geq 0$ and $t - s < 0$
Thus, $(t_n - s_n) - (t - s) > 0$
Let $0 < \epsilon < (t_n - s_n) - (t - s)$.
Then, $|(t_n - s_n) - (t - s)| > \epsilon$. A contradiction is reached.
Thus, $s \leq t$

2 9.15

$\frac{s_{n+1}}{s_n} = \frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a}{n!}} = \frac{a}{n+1}$
$\lim \frac{s_{n+1}}{s_n} = 0$
$\forall \epsilon > 0, \exists N\, s.t. \forall n > N, |\frac{s_{n+1}}{s_n}| < \epsilon$
Let $\epsilon = \gamma < 1$
$\frac{|s_{n+1}|}{|s_n|} < \gamma$

$|s_{n+1}| < \gamma|s_n| \Rightarrow |s_n| < \gamma|s_{n-1}| < \gamma^2|s_{n-2}| < ... < \gamma^{n-N}|s_N|$
$\lim \gamma^{n-N}|s_N| = \lim \gamma^n \gamma^{-N}|s_N| = 0$ (because $\gamma < 1$)
Since $|s_n| \leq \gamma^{n-N}|s_N|$, $\lim |s_n| \leq \lim \gamma^{n-N}|s_N| = 0$ from 9.9c.
Thus, $\lim s_n = 0$

3 10.7

Let $s_1 = \sup S - \frac{1}{m}$ s.t. $m \in \mathbf{N}$ and $s_1 \in S$. This is possible because by the definition of sup, $\exists s \in S$ s.t. $s > \sup S - \epsilon$, where $\epsilon > 0$.
Define $s_n = \sup S - \frac{1}{m+n-1}$. To show that $s_n \in S$, for the sake of contradiction assume that $s_n \notin S$. We know s_n is a monotonically increasing sequence, because $\frac{1}{m+n-1}$ is monotonically decreasing. If $s_n \notin S$, then $s_n \geq \sup S$ by the definition of $\sup S$ being the smallest upper bound for S. This is a contradiction to $s_n = \sup S - \frac{1}{m+n-1} < \sup S$
Therefore, $\lim s_n = \sup S$.

4 10.8

WTS: $\forall n, \sigma_{n+1} \geq \sigma_n$
$\sigma_{n+1} = \frac{1}{n+1}(s_1 + s_n + s_{n+1}) = \frac{1}{n+1}(\frac{n(s_1+...+s_n)}{n} + s_{n+1}) = \frac{1}{n+1}(n\sigma_n + s_{n+1})$
We can show $ns_{n+1} = s_{n+1} + ... + s_{n+1} \geq s_1 + ... + s_n$ by induction.
Thus, $s_{n+1} \geq \frac{s_1+...+s_n}{n} = \sigma_n$
Thus, $n\sigma_n + s_{n+1} \geq n\sigma_n + \sigma_n \Rightarrow n\sigma_n + s_{n+1} \geq (n+1)\sigma_n \Rightarrow \frac{1}{n+1}(n\sigma_n + s_{n+1}) = \sigma_{n+1} \geq \sigma_n$

5 10.9

5.1 a

$s_2 = (\frac{1}{2})1^2 = \frac{1}{2}$
$s_3 = (\frac{2}{3})(\frac{1}{2})^2 = \frac{1}{6}$
$s_4 = (\frac{3}{4})(16)^2 = \frac{1}{48}$

5.2 b

WTS: s_n is decreasing and bounded, by showing $0 < s_{n+1} \leq 1$
Proof by induction:
The above inequality holds for $0 < s_1 = 1 \leq 1$ and $0 < s_2 = \frac{1}{2} \leq 1$
Assume it holds for some $n \geq 2$
WTS: $(\frac{n}{n+1})s_n^2 < s_n$
$ns_n^2 < (n+1)s_n$
$ns_n < n + 1$
$s_n < \frac{n+1}{n} = 1 + \frac{1}{n}$
We know $s_n \leq 1 < 1 + \frac{1}{n}$

WTS: $0 < s_{n+1}$
$0 < (\frac{n}{n+1})s_n^2$
This is true because $n > 0$ and $s_n^2 > 0$

5.3 c

Let $\lim s_n = s$.
s satisfies $ns^2 = (n+1)s$
$ns^2 - (n+1)s = 0$
$s(ns - n - 1) = 0$
$s = 0$ since $s \neq \frac{n+1}{n}$, because $s_n < 1\,\forall n$

6 10.10

6.1 a

$s_2 = \frac{1}{3}(1+1) = \frac{2}{3}$
$s_3 = \frac{1}{3}(\frac{2}{3}+1) = \frac{5}{9}$
$s_4 = \frac{1}{3}(\frac{5}{9}+1) = \frac{14}{27}$

6.2 b

$s_1 = 1 > \frac{1}{2}$
Assume $s_n > \frac{1}{2}$ for some $n \geq 1$
$s_{n+1} = \frac{1}{3}(s_n+1) = \frac{1}{3}s_n + \frac{1}{3} > \frac{1}{3}(\frac{1}{2}) + \frac{1}{3} = \frac{1}{2}$

6.3 c

WTS: $s_{n+1} \leq s_n$
$s_n > \frac{1}{2} \Rightarrow \frac{2}{3}s_n > \frac{1}{3}$
$s_{n+1} = \frac{1}{3}(s_n+1) = \frac{1}{3}s_n + \frac{1}{3} < \frac{1}{3}s_n + \frac{2}{3}s_n = s_n$

6.4 d

s_n is bounded and decreasing $\Rightarrow \lim s_n$ exists
Let $\lim s_n = s$
s satisfies $s = \frac{1}{3}(s+1)$
$\frac{2}{3}s = \frac{1}{3}$
$s = \frac{1}{2}$

7 10.11

7.1 a

Show t_n is bounded
Prove by induction that $\forall n, t_n > 0$

$t_1 = 1 > 0$
Assume $t_n > 0$
WTS: $t_{n+1} = (1 - \frac{1}{4(n+1)^2})t_n > 0$
$(1 - \frac{1}{4(n+1)^2}))$ is positive $\forall n$, and $t_n > 0$. Thus $t_{n+1} > 0$

Show t_n is decreasing
WTS: $t_{n+1} \leq t_n$
$(1 - \frac{1}{4n^2})t_n \leq t_n$
$(4n^2 - 1)t_n \leq 4n^2 t_n$
$t_n \geq -1$
Because $t_n > 0, t_n \geq -1$

Because t_n is decreasing and bounded, $\lim t_n$ exists

7.2 b

$\frac{1}{e}$?

8 Squeeze test

Let $\epsilon > 0$
$\lim a_n = L \Rightarrow \exists N_0$ s.t. $\forall n > N_0, |L - a_n| < \epsilon$
$a_n > L - \epsilon$
$\lim c_n = L \Rightarrow \exists N_1$ s.t. $\forall n > N_1, |L - c_n| < \epsilon$
$c_n < L + \epsilon$
Let $N > \max N_0, N_1$
$\forall n > N, a_n < b_n < c_n \Rightarrow L - \epsilon < b_n < L + \epsilon \Rightarrow |b_n - L| < \epsilon \Rightarrow \lim b_n = L$