Math 104 HW 3

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1 10.6

1.1 a

Let $\epsilon > 0$ and $N = \log_2(\frac{1}{\epsilon}) + 1$. Then $n > N \Rightarrow n - 1 > \log_2(\frac{1}{\epsilon}) \Rightarrow \epsilon > 2^{1-n}$
Let $m, n > N$. $|s_n - s_m| < \sum_{i=m}^{n-1} 2^{-i} < \sum_{i=m}^{\infty} 2^{-i} = 2^{1-m} < \epsilon$
Thus, (s_n) is Cauchy and convergent.

1.2 b

No, it is possible for (s_n) to be unbounded. If we use an analogous proof to part a, we can only bound $|s_n - s_m| < \sum_{i=m}^{n-1} \frac{1}{i} < \sum_{i=m}^{\infty} \frac{1}{i} < \infty$. Thus, for any given ϵ, one can pick $n > N$ s.t. $|s_n - s_m| > \epsilon$.

2 11.2

2.1 a

a_n: $n_k = 2k$. $a_{n_k} = 1$
b_n: $n_k = k$. b_n is monotonically decreasing.
c_n: $n_k = k$. c_n is monotonically increasing.
d_n:$n_k = k$. d_n is monotonically decreasing.

Proof that d_n is monotonically decreasing:

$d_{n+1} = \frac{6(n+1)+4}{7(n+1)-3} = \frac{6n+5}{7n+4} \leq \frac{6n+4}{7n-3}$ if $n > \frac{1}{35}$, which is true.

2.2 b

a_n: $\{1, -1\}$
b_n: $\{0\}$
c_n: $\{+\infty\}$
d_n: $\{\frac{6}{7}\}$

2.3 c

$\limsup a_n = 1, \liminf a_n = -1$
$\limsup b_n = 0, \liminf b_n = 0$
$\limsup c_n = +\infty, \liminf c_n = +\infty$
$\limsup d_n = \frac{6}{7}, \liminf d_n = \frac{6}{7}$

2.4 d

a_n: diverges
b_n: converges
c_n: diverges to $+\infty$
d_n: converges

2.5 e

a_n: bounded
b_n: bounded
c_n: unbounded
d_n: bounded

3 11.3

3.1 a

s_n: $n_k = 6k$. $s_{n_k} = 1$
t_n: $n_k = k$. t_n is monotonically decreasing.
u_n: $n_k = 2k$. $u_{n_k} = \frac{1}{2^k}$
v_n:$n_k = 2k$. $v_{n_k} = \frac{1}{2k}$

3.2 b

s_n: $\{\frac{1}{2}, -\frac{1}{2}, -1, 1\}$
t_n: $\{0\}$
u_n: $\{0\}$
v_n: $\{1, -1\}$

3.3 c

$\limsup s_n = 1, \liminf s_n = -1$
$\limsup t_n = 0, \liminf t_n = 0$
$\limsup u_n = 0, \liminf u_n = 0$
$\limsup v_n = 1, \liminf v_n = -1$

3.4 d

s_n: diverges
t_n: converges
u_n: converges
v_n: diverges

3.5 e

s_n: bounded
t_n: bounded
u_n: bounded
v_n: bounded

4 11.5

4.1 a

The set of subsequential limits S consists of all rational numbers in the interval $[0, 1]$.

Proof that any rational $r \in [0, 1]$ is a subsequential limit of (q_n):

WTS: The set $\{n : |q_n - r| < \epsilon\}$ is infinite.
By the denseness of $\mathbb{Q}$, there exist an infinitely many rationals between a and b for $a, b \in \mathbb{R}$ (Ex 4.11).
Thus, there exist infinitely many rationals between $r - \epsilon$ and $r + \epsilon$, and because q_n consists of all rationals in $(0, 1]$, $\{n : |q_n - r| < \epsilon\}$ is infinite.

4.2 b

By Thm 11.8, $\sup S = \limsup s_n$ and $\inf S = \liminf s_n$. Thus, $\limsup s_n = 1$ and $\liminf s_n = 0$.

5 What is limsup?

Take the tail of a sequence, then consider the group of values within the tail of the sequence. Limsup is the smallest upper bound of this group of values. Sup is the smallest upper bound for a set, but limsup is applied to a sequence rather than a set. It is counter-intuitive that it is possible for the limsup to converge but not the limit of the sequence itself (for example, $s_n = (-1)^n$ has $\limsup s_n = 1$, but s_n does not converge).