Math 104 HW 4

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1 12.10

Prove (s_n) bounded $\Rightarrow \limsup |s_n| < +\infty$:

(s_n) bounded $\Rightarrow (|s_n|)$ bounded.
Thus, $\forall N$, $\sup\{|s_n| : n > N\} < M$ for some constant M. Thus, $\limsup |s_n| < +\infty$.

Prove (s_n) not bounded $\Rightarrow \limsup |s_n| = +\infty$:

If (s_n) is not bounded, $(|s_n|)$ is also not bounded.
$(|s_n|)$ consists of all non-negative numbers. Thus, $\lim |s_n| = +\infty \Rightarrow \limsup |s_n| = +\infty$

2 12.12

2.1 a

We know $\liminf s_n \leq \limsup s_n$ and $\liminf \sigma_n \leq \limsup \sigma_n$

Let $n > M$
$\sigma_n = \frac{1}{n}(s_1 + ... + s_n) \leq \frac{1}{n}(s_1 + ... + s_N + s_{N+1} + ... + s_n)$
$\leq \frac{1}{M}(s_1 + ... + s_N) + \frac{1}{n}(s_{N+1} + ... + s_n)$
$\leq \frac{1}{M}(s_1 + ... + s_N) + \frac{n-N}{n} \sup\{s_i\}_{i=N+1}^{n}$
$\leq \frac{1}{M}(s_1 + ... + s_N) + \sup\{s_i\}_{i=N+1}^{\infty}$
$\Rightarrow \sup\{\sigma_n : n > M\} \leq \frac{1}{M}(s_1 + ... + s_N) + \sup\{s_i : n > N\}$
Take $M \to \infty$: $\limsup \sigma_n \leq \sup\{s_i : n > N\}$
Take $N \to \infty$: $\limsup \sigma_n \leq \limsup s_n$

Let $n > N$
$\sigma_n = \frac{1}{n}(s_1 ... + s_n)$
$= \frac{1}{n}(s_1 ... + s_N) + \frac{1}{n}(s_{N+1} ... + s_n)$
$\geq \frac{1}{n}(s_1 + ... + s_N) + \frac{n-N}{n} \inf\{s_n\}_{i=N+1}^{n}$
$\geq \frac{1}{n}(s_1 + ... + s_N) + \frac{n-N}{n} \inf\{s_n\}_{i=N+1}^{\infty}$

$\Rightarrow \liminf \sigma_n \geq \liminf \frac{1}{n}(s_1 + ... + s_N) + \liminf \frac{n-N}{n} \inf\{s_n\}_{i=N+1}^{\infty}$
$= \inf\{s_n\}_{i=N+1}^{\infty}$
Take $N \to \infty$: $\liminf \sigma_n \geq \liminf s_n$

2.2 b

If $\lim s_n$ exists, then $\lim s_n = \limsup s_n = \liminf s_n \Rightarrow \liminf s_n = \liminf \sigma_n = \limsup \sigma_n = \limsup s_n \Rightarrow \lim \sigma_n$ exists and $\lim s_n = \lim \sigma_n$

2.3 c

Let $s_n = (-1)^n$. Then $\sigma_n = \frac{1}{n}$ if n is odd and 0 if n is even. $\limsup s_n = 1$ and $\liminf s_n = -1$ so $\lim s_n$ does not exist, but $\lim \sigma_n = 0$

3 14.2

3.1 a

Diverges.
Using the comparison test, $\frac{n-1}{n^2} = \frac{1}{n} - \frac{1}{n^2}$. Because $\sum \frac{1}{n}$ diverges, $\sum \frac{n-1}{n^2}$ diverges.

3.2 b

Diverges.
The partial sums are $s_n = -1$ if n is odd and 0 if n is even. The partial sums diverge, thus the series diverges.

3.3 c

Converges.
$\frac{3n}{n^3} = \frac{3}{n^2}$ which converges because $\frac{1}{n^2}$ converges.

3.4 d

Converges.
Using the ratio test, $\lim |fraca_{n+1}a_n| = \lim \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim \frac{(n+1)^3}{3n^3} = \frac{1}{3} < 1$

3.5 e

Converges.
Using the ratio test, $\lim |\frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}}| = \lim |\frac{n+1}{n^2}| = 0 < 1$

3.6 f

Converges.
Using the root test, $\lim |\frac{1}{n^n}|^{\frac{1}{n}} = \lim |\frac{1}{n}| = 0 < 1$

3.7 g

Converges.
Using the ratio test, $\lim \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim \frac{(n+1)}{2n} = \frac{1}{2} < 1$

4 12.10

Let $a_n = 2^{(-1)^n+n}$.
For even n, $\frac{a_{n+1}}{a_n} = \frac{2^{n+2}}{2^{n-1}} = 8$ and for odd n, $\frac{2^n}{2^{n+1}} = \frac{1}{2}$. Thus, $\frac{1}{2} = \liminf \frac{|a_{n+1}|}{|a_n|} < 1 < \limsup \frac{|a_{n+1}|}{|a_n|} = 8$ and the ratio test gives no information.

$\lim |a_n|^{\frac{1}{n}} = 2^{\frac{1}{n}+1}$ for even n, and $2^{-\frac{1}{n}+1}$ for odd n. Thus, $\lim |a_n|^{\frac{1}{n}} = 2 > 1$, and $\sum a_n$ diverges by the root test.

5 6

5.1 a

$s_n = \sqrt{n+1} - 1$. (s_n) diverges $\Rightarrow \sum a_n$ diverges.

5.2 b

$s_n = -\sqrt{1} + \sum_{k=2}^{n} \sqrt{k}(\frac{1}{k} - \frac{1}{k+1}) + \frac{\sqrt{n+1}}{n}$.
$\sqrt{k}(\frac{1}{k} - \frac{1}{k+1}) = \frac{1}{k^{\frac{3}{2}} + k^{\frac{1}{2}}}$
$\sum \frac{1}{k^{\frac{3}{2}} + k^{\frac{1}{2}}} \leq \sum \frac{1}{k^{\frac{3}{2}}} \Rightarrow \sum \sqrt{k}(\frac{1}{k} - \frac{1}{k+1})$ converges $\Rightarrow (s_n)$ converges $\Rightarrow \sum a_n$ converges.

5.3 c

Using the root test, $\lim |a_n|^{\frac{1}{n}} = \lim n^{\frac{1}{n}} - 1 = 0$. Thus, $\sum a_n$ converges.

5.4 d

Not sure about complex numbers

6 7

Let $\epsilon > 0$. Let $n > m > N$.
$|\frac{\sqrt{a_m}}{m} + ... + \frac{\sqrt{a}}{n}| \leq \frac{\sqrt{a_m}}{m} + ... + \frac{\sqrt{a_n}}{n} = \frac{1}{m}(\sqrt{a_m} + ... + \sqrt{a_n})$

Set N large enough such that $a_n < \frac{m\epsilon}{n-m+1} \forall n > N$, which is possible because $\sum a_n$ converges $\Rightarrow \lim a_n = 0$.
Thus, $|\frac{\sqrt{a_m}}{m} + ... + \frac{\sqrt{a}}{n}| \leq \frac{1}{m}(m\epsilon) = \epsilon$

7 9

7.1 a

$\limsup |n^3|^{\frac{1}{n}} = \lim (n^{\frac{1}{n}})^3 = 1$
$R = 1$

7.2 b

$\limsup |\frac{2^n}{n!}|^{\frac{1}{n}} = \lim \frac{2}{(n!)^{\frac{1}{n}}}$
Let $a_n = \frac{1}{n!}$
$\limsup \frac{a_{n+1}}{a_n} = \limsup \frac{1}{n+1} = 0$
$\liminf \frac{a_{n+1}}{a_n} = 0$
Thus, $\lim a_n^{\frac{1}{n}} = 0 \Rightarrow \lim \frac{1}{(n!)^{\frac{1}{n}}} = 0 \Rightarrow R = \infty$

7.3 c

$\limsup |\frac{2^n}{n^2}|^{\frac{1}{n}} = \lim \frac{2}{(n^{\frac{1}{n}})^2} = 2 \Rightarrow R = \frac{1}{2}$

7.4 d

$\limsup |\frac{n^2}{3^n}|^{\frac{1}{n}} = \lim \frac{(n^{\frac{1}{n}})^3}{3} = \frac{1}{3} \Rightarrow R = 3$

8 11

8.1 a

$\lim \frac{a_n}{1+a_n} = \lim \frac{1}{1+\frac{1}{a_n}} > 0$, since $\lim a_n \geq 0$
Thus, $\sum \frac{a_n}{1+a_n}$ is divergent.

8.2 b

$\frac{a_{N+1}}{s_{N+1}} + ... + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1}}{s_{N+k}} + ... + \frac{a_{N+k}}{s_{N+k}} = \frac{a_{N+1}+...+a_{N+k}}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}$
$\forall N$, $\exists k$ s.t. $\frac{a_{N+1}}{s_{N+1}} + ... + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}} > \epsilon$
Thus, $\sum \frac{a_n}{s_n}$ is not Cauchy $\Rightarrow \sum \frac{a_n}{s_n}$ diverges

8.3 c

Since $s_{n-1} < s_n$, $\frac{a_n}{s_n^2} \leq \frac{a_n}{s_{n-1}s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{1}{s_{n-1}} - \frac{1}{s_n}$
Because $\sum a_n$ diverges, $\sum \frac{1}{s_{n-1}}$ and $\sum \frac{1}{s_n}$ converge. Thus, $\sum \frac{a_n}{s_n^2}$ converges by comparison test.

8.4 d

$\frac{a_n}{1+na_n}$ converges or diverges.

$\frac{a_n}{1+n^2a_n} \leq \frac{a_n}{n^2a_n} = \frac{1}{n^2}$
Because $\sum \frac{1}{n^2}$ converges, $\sum \frac{a_n}{1+n^2a_n}$ converges.