Math 104 HW 5

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1 13.3

1.1 a

$|x-y| \geq 0 \ \forall x, y$, so $\sup\{|x_j - y_j| : j = 1, 2, ...\} \geq 0$
If $x = y$, $d(x, x) = \sup\{0, 0, ..., 0\} = 0$
If $d(x, y) = 0$, because $|x_j - y_j| \geq 0 \ \forall j$,
$\sup\{|x_j - y_j| : j = 1, 2, ...\} = 0 \Rightarrow |x_j - y_j| = 0 \ \forall j$
$\Rightarrow x = y$

$d(x, y) = d(y, x)$ because $|x_j - y_j| = |y_j - x_j|$

$d(x, y) + d(y, z) = \sup\{|x_j - y_j| : j = 1, 2, ...\} + \sup\{|y_j - z_j| : j = 1, 2, ...\}$
$= \sup\{|x_j - y_j| + |y_j - z_j| : j = 1, 2, ...\}$
$\geq \sup\{|x_j - z_j| : j = 1, 2, ...\} = d(x, z)$

1.2 b

No, the metric can produce a value that is not real.

Example: $x = (0, 0, ...), y = (1, 1, ...), d(x, y) = \infty$

2 13.5

2.1 a

$x \in \bigcap\{S \backslash U : U \in \mathcal{U}\}$
$\iff x \in \{S\} \ \forall U \in \mathcal{U}$
$\iff x \notin U \ \forall U \in \mathcal{U}$
$\iff x \notin \bigcup\{U : U \in \mathcal{U}\}$
$\iff x \in S \backslash \bigcup\{U : U \in \mathcal{U}\}$

$\bigcap\{S \backslash U : U \in \mathcal{U}\} = S \backslash \bigcup\{U : U \in \mathcal{U}\}$

2.2 b

Using the notation from part a, let $\mathcal{U}$ be a family of open sets. Then $\bigcap\{S \backslash U : U \in \mathcal{U}\}$ is the intersection of a collection of closed sets. By part a, the intersection of closed sets is equivalent to $S \backslash \bigcup\{U : U \in \mathcal{U}\}$. Because the union of open sets is also open, $S \backslash \bigcup\{U : U \in \mathcal{U}\}$ is closed.

3 13.7

Any set in $\mathbb{R}$ consists of a disjoint union of intervals of the form $[a, b], (a, b), (a, b]$, and $[a, b)$. We also allow a and b to take on values $+\infty$ and $-\infty$. We show that if a set in $\mathbb{R}$ is open $[a, b], [a, b), (a, b]$ cannot be part of that disjoint union. Given interval $[a, b)$, consider a. There does not exist any open ball $B_r(a)$, because for any $r > 0$, $a - r \notin [a, b)$. The same argument can be applied for intervals of form $(a, b]$ and $[a, b]$. Thus, any set in $\mathbb{R}$ can only consist of a disjoint union of intervals of the form (a, b). To show this disjoint union of open intervals is countable, for each interval (a, b), there exists a rational $q \in (a, b)$ by the Denseness of $\mathbb{Q}$. Thus, the mapping from open intervals to $\mathbb{Q}$ is injective, so the set of open intervals is countable.

4 4

WTS: $\forall \bar{p} \in \bar{\bar{S}}$, $\bar{p} \in S$, i.e. there exists a sequence $(p_n) \in X$ s.t. $p_n \to p$

Let $\epsilon > 0$

Because $\bar{p} \in \bar{\bar{S}}$, there exists a sequence $(\bar{p}_n)$ s.t. $\bar{p}_n \to \bar{p}$
Thus, $\exists N_1 > 0$ s.t. $\forall n_1 > N_1$, $|\bar{p}_{n_1} - \bar{p}| > \frac{\epsilon}{2}$

Fix n_1. Because $\bar{p}_{n_1} \in \bar{S}$, $\exists N$ s.t. $\forall n_2 > N$ s.t. $|p_{n_2} - \bar{p}_{n_1}| < \frac{\epsilon}{2}$

Thus, $\forall n_2 > N$, $|p_{n_2} - \bar{p}| \leq |p_{n_2} - \bar{p}_{n_1}| + |\bar{p}_{n_1} - \bar{p}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$
Thus, there exists a sequence $(p_n) \in X$ s.t. $p_n \to p$

5 5

WTS: $\bar{S} \subset \bigcap\{F \subset X \text{ closed}, S \subset F\}$

Let $s \in \bar{S}$. Thus, there exists a sequence $(s_n) \in S$ s.t. $s_n \to s$
Pick any F s.t. $S \subset F$ and F is closed.
Because $S \subset F$, $(s_n) \in F$.
F is closed, so all sequences $(f_n) \in F$ have $f_n \to f \in F$
Thus, $s \in F$
Because we picked arbitrary F, $s \in$ all F's s.t. $S \subset F$ and F is closed. Thus,

$s \in \bigcap\{F \subset X \text{ closed}, S \subset F\}$

To show $\bar{S} = \bigcap\{F \subset X \text{ closed}, S \subset F\}$, let one such F be $\bar{S}$. $S \subset \bar{S}$, and $\bar{S}$ is closed.
Thus, $\bigcap\{F \subset X \text{ closed}, S \subset F\} = \bar{S}$