Math 104 HW 6

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1 1

Let (x_n, y_n) be a sequence in $[0,1]^2$. The sequence (x_n) is bounded, so there exists a subsequence (x_{n_k}) that converges to some $x \in [0,1]$ (since $[0,1]$ is closed). Let (x_{n_k}, y_{n_k}) be the subsequence of (x_n, y_n) containing (x_{n_k}). By the same argument, there exists a subsequence $y_{n_{k_l}}$ that converges to some $y \in [0,1]$. $(x_{n_{k_l}}, y_{n_{k_l}})$ is still a subsequence of (x_n, y_n) and $(x_{n_{k_l}}, y_{n_{k_l}})$ converges to (x, y) (any subsequence of (x_{n_k}) still converges to x); thus, $[0,1]^2$ is sequentially compact.

2 2

E is uncountable, which can be shown using Cantor's diagonalization argument. Assume by contradiction that E is countable. Then, the set of decimal expansions that are infinite in E is countable, and these decimal expansions can be listed. For the nth decimal point of point n, change the digit (if the decimal point is 4, change it to 7 and vice versa). By construction, this new decimal expansion is in E, but is not enumerated in the list. Thus, a contradiction exists, and E is uncountable.

First, we show that E is closed. We prove that for $(p_n) \in E$, if (p_n) converges to p, $p \in E$. For the sake the contradiction, suppose $(p_n) \in E$ and $p_n \to p$ s.t. $p \notin E$. Then, the decimal expansion for p consists of at least one digit that is not 4 nor 7. p can then be represented as $0.**...*x*...$, where $x \neq 4, 7$ and $*$ is any digit. Consider the closest element to p in (p_n). Call this element p_i. Set $\epsilon = |p - p_i|$. Then, there does not exist N s.t. $\forall n > N$, $|p_n - p| < \epsilon$, which means (p_n) does not converge to p and a contradiction is reached.

E is compact. Let (p_n) be a sequence in E. (p_n) is bounded, so there exists a subsequence of (p_n) that converges to some $p \in E$, because E is closed. Thus, E is compact.

3 3

Yes, it is possible that the inclusion is a strict inclusion. Consider if all subsets A_i are all equal and equal $(0, 1)$. Then $B = \cup A_i = A_1$. $\bar{A}_i = [0, 1]$, and $\cup \bar{A}_i = \bar{A}_1 = \bar{B}$.

4 4

The argument is wrong where it states ”adjacent open intervals sandwich a closed interval.” The set $\mathbb{R}$ is closed (but does not consist of a countable union of closed intervals), and the complement of $\mathbb{R}$ is the empty set, which is open. However, there are zero open intervals in the complement of $\mathbb{R}$, so one cannot use the argument that the closed intervals are sandwiched by countably infinite open intervals.