Math 104 HW 7

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March 13, 2022

1 1

Let $X \times Y \subset \cup_{\alpha \in I} U_\alpha$, where each U_α is open

$\forall (x, y) \in X \times Y$, $(x, y) \in U_\alpha$
Because U_α is open, it is possible to construct an open box $B_{(x,y)} \in U_\alpha$ around (x, y) of the form $V_{(x,y)} \times W_{(x,y)}$, where $V_{(x,y)} \subset X$ and $W_{(x,y)} \subset Y$.

Then, $X \subset \cup_{(x,y) \in X \times Y} V_{(x,y)}$, $Y \subset \cup_{(x,y) \in X \times Y} W_{(x,y)}$
Because X and Y are open cover compact, $X \subset \cup_{i=1}^{n} V_{(x_i,y_i)}$ and $Y \subset \cup_{i=1}^{m} W_{(x_i,y_i)}$

$\forall (x, y)$, pick U_α s.t. $x \in V_{(x_i,y_i)}$ and $y \in W_{(x_j,y_j)}$ and $V_{(x_i,y_i)} \times W_{(x_j,y_j)} \in U_\alpha$
Because $\cup_{i=1}^{n} V_{(x_i,y_i)}$ and $\cup_{i=1}^{m} W_{(x_i,y_i)}$ consist of a finite union of sets, $V_{(x_i,y_i)} \times W_{(x_j,y_j)}$ corresponds to finitely many (at most $n \times m$) open sets U_α.

Thus, $X \times Y$ can be covered by a finite subcover of $\cup_{\alpha \in I} U_\alpha$.

2 2

2.1 a

False, consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1$. Let $A = (0, 1)$. Then $f(A) = [1, 1]$, which is closed.

2.2 b

False, consider $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \frac{x}{x+1}$. Let $A = [0, +\infty)$, which is closed. Then $f(A) = [0, 1)$, which is not closed.

2.3 c

False, consider $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Let $A = (0, 1]$. Then $f(A) = [1, +\infty]$, which is unbounded.

2.4 d

True. Let $f(A) = \cup_{\alpha \in I} U_\alpha$ where each U_α is open.
$\forall \alpha \in I$, $f(f^{-1}(U_\alpha)) \subset U_\alpha \Rightarrow A \subset \cup_{\alpha \in I} f^{-1}(U_\alpha)$
Because f is continuous and U_α, $f^{-1}(U_\alpha)$ is open.
Thus, because A is compact, $A \subset f^{-1}(U_1) \cup ... \cup f^{-1}(U_N)$, and $f(A) \subset U_1 \cup ... \cup U_N$. Thus $f(A)$ is compact.

2.5 e

True. WTS: $f(A)$ is connected, i.e. $f(A)$ cannot be written as $G \sqcup H$ where G and H are 2 non-empty open subsets of $f(A)$.

Proof by contradiction:
Let $f(A) = G \sqcup H$ where where G and H are 2 non-empty open subsets of $f(A)$
Because f is continuous, $f^{-1}(G)$ and $f^{-1}(H)$are open in X
$A \subset f^{-1}(G) \sqcup f^{-1}(H)$
$A = A \cap (f^{-1}(G) \sqcup f^{-1}(H))$
$A = (A \cap f^{-1}(G)) \sqcup (A \cap f^{-1}(H))$
Because $f^{-1}(G)$ is open in X, $A \cap f^{-1}(G)$ is open in A
Thus, A can be written as a disjoint union of 2 non-empty open subsets of A, which is a contradiction, because A is connected. Thus, $f(A)$ is connected.

3 3

Suppose that f is surjective and continuous. Then from 2d, if $A \subset [0, 1]$ is compact, then $f(A)$ is compact. $[0, 1]$ is compact, and and $f([0, 1]) = \mathbb{R}$ because f is surjective. However, $\mathbb{R}$ is not compact. Thus, there is a contradiction and f cannot be both surjective and continuous.