Math 104 HW 8

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1 1

We show that f_n converges uniformly to $f(x) = \frac{1}{2}$

Let $\epsilon > 0$
$\forall x \in \mathbb{R}$, $|\frac{n+\sin x}{2n+\cos(n^2x)} - \frac{1}{2}| = |\frac{2\sin x - \cos(n^2x)}{4n+2\cos(n^2x)}| \leq |\frac{3}{4n+2\cos(n^2x)}| \leq |\frac{3}{4n-2}|$
There exists an $N > 0$ s.t. $\forall n > N$, $|\frac{3}{4n-2}| < \epsilon$
Thus, f_n converges uniformly to $f(x) = \frac{1}{2}$

2 2

$f_n(x) = a_n x^n$ is continuous, so $\sum_{n=1}^{\infty} f_n(x)$ is continuous
WTS: $f_n \to f$ uniformly

Let $\epsilon > 0$
$\forall x \in [-1,1]$, $|a_n x^n + a_{n+1}x^{n+1} + ... + a_m x^m| \leq ||a_n x^n| + |a_{n+1}x^{n+1}| + ... + |a_m x^m|| \leq ||a_n| + |a_{n+1}| + ... + |a_m||$
Because $\sum |a_n|$ is convergent, $\exists N$ s.t. $\forall n, m > N$, $||a_n| + |a_{n+1}| + ... + |a_m|| < \epsilon$
Thus $\forall x \in [-1,1]$, $\exists N$ s.t. $\forall n, m > N$, $|a_n x^n + a_{n+1}x^{n+1} + ... + a_m x^m| < \epsilon$
Thus, $f_n \to f$ uniformly, and f is continuous

$\sum_{n=1}^{\infty} |\frac{1}{n^2}|$ converges, so $\sum_{n=1}^{\infty} |\frac{1}{n^2}| < \infty$
Thus, $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is continuous on $[-1,1]$

3 3

$\forall x \in [-a,a], \forall n$, $|f_n(x)| = |x^n| \leq a^n$
$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} < \infty$
Thus, $f(x) = \sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent in $[-a,a]$.
$\forall x \in (-1,1)$, it is possible to pick a s.t. $x \in [-a,a]$. Since f is uniformly convergent on $[-a,a]$, f is continuous on $[-a,a]$, and f is continuous at x. Thus, f is continuous on $(-1,1)$

$|f_n(x) - f(x)| = |\sum_{i=0}^{n} x^{-} \frac{1}{1-x}| = |\frac{1-x^{n+1}}{1-x} - \frac{1}{1-x}| = |\frac{x^{n+1}}{x-1}|$
To show f is not uniformly convergent on $(-1, 1)$, we show that for $\epsilon > 0, \forall n > 0$, $\exists x$ s.t. $|\frac{x^{n+1}}{x-1}| \geq \epsilon$. Thus, there does not exist $N > 0$ s.t. $\forall \epsilon > 0, \forall n > N$, $|f_n(x) - f(x)| < \epsilon$

Let $\epsilon = \frac{1}{3}$
WTS: $|\frac{x^{n+1}}{x+1}| \geq \frac{1}{3} \iff |x^{n+1}| \geq \frac{1}{3}|x - 1|$
$\frac{1}{3}|x - 1| \leq \frac{2}{3}$
It is possible to choose $x \in (-1, 1)$ s.t. $|x^{n+1}| \geq \frac{2}{3}$ by choosing x arbitrarily close to 1
Thus, f is not uniformly convergent on $(-1, 1)$