Math 104 HW 9

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1

Let

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{2}e^{2-\frac{1}{x}} & \text{if } 0 < x \leq \frac{1}{2} \\ -\frac{1}{2}e^{2-\frac{1}{1-x}} & \text{if } \frac{1}{2} < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

WTS: for $g(x) = \frac{1}{2}e^{2-\frac{1}{1-x}} + 1, g^{(n)}(1) = 0$ for $n = 1, 2, ...$
can show $g^{(n)}(1) = 0$ for $n = 1, 2, ...$ where $g(x) = e^{-\frac{1}{1-x}}$
Let $y = 1 - x$, then $g(y) = e^{-\frac{1}{y}}$
We know $g^{(n)}(0) = 0$ for $n = 1, 2, ...$
$y = 0 \Rightarrow x = 1$
Thus, $g^{(n)}(1) = 0$ for $n = 1, 2, ...$

WTS: for $g(x) = \frac{1}{2}e^{2-\frac{1}{x}}$ and $h(x) = \frac{1}{2}e^{2-\frac{1}{1-x}} + 1$, $g^{(n)}(\frac{1}{2}) = h^{(n)}(\frac{1}{2})$ for $n = 1, 2, ...$
Let $h(x) = -g(1 - x) + 1$, then $h'(x) = g'(1 - x)$
At $x = \frac{1}{2}$, $g'(x) = g'(\frac{1}{2}) = g'(1 - \frac{1}{2}) = h'(\frac{1}{2})$
For odd n, $g^{(n)}(\frac{1}{2}) = h^{(n)}(\frac{1}{2})$

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Let $g(x) = C_0x + \frac{C_1}{2}x^2 + ... + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1}$
Then, $g'(x) = f(x)$
$g(0) = 0$
$g(1) = C_0 + \frac{C_1}{2} + ... + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$
By Rolle Thm, $\exists x \in (0, 1)$ s.t. $g'(x) = 0$
Thus, $C_0 + C_1x + ... + C_{n-1}x^{n-1} + C_nx^n = 0$ has at least one real root between 0 and 1

3

Let $\epsilon > 0$
By MVT, $\exists c \in (x, t)$ s.t. $f'(c) = \frac{f(t)-f(x)}{t-x}$
$|\frac{f(t)-f(x)}{t-x} - f'(x)| = |f'(c) - f'(x)|$
f' is continuous $\Rightarrow \exists \delta$ s.t. if $|t-x| < \delta$, $|f'(t) - f'(x)| < \epsilon$
Because $c \in (x, t)$, $|c - x| < \delta$
Thus, $|f'(c) - f'(x)| < \epsilon$

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By induction,
$f^{(1)}(t) = (t-\beta)Q^{(1)}(t) + Q(t) \Rightarrow Q(\alpha) = f^{(1)}(\alpha) + Q^{(1)}(\alpha)(\beta - \alpha)$
$f^{(2)}(t) = (t-\beta)Q^{(2)}(t) + 2Q^{(1)}(t) \Rightarrow Q^{(1)}(\alpha) = \frac{1}{2}(f^{(2)}(\alpha) + Q^{(2)}(\alpha)(\beta - \alpha))$
$f^{(3)}(t) = (t-\beta)Q^{(3)}(t) + 3Q^{(2)}(t) \Rightarrow Q^{(2)}(\alpha) = \frac{1}{3}(f^{(3)}(\alpha) + Q^{(3)}(\alpha)(\beta - \alpha))$
$\vdots$
$f^{(n)}(t) = (t-\beta)Q^{(n)}(t) + nQ^{(n-1)}(t) \Rightarrow Q^{(n-1)}(\alpha) = \frac{1}{n}(f^{(n)}(\alpha) + Q^{(n)}(\alpha)(\beta - \alpha))$

$f(\beta) = f(\alpha) + (\beta - \alpha)Q(\alpha)$
$= f(\alpha) + (\beta - \alpha)f^{(1)}(\alpha) + Q^{(1)}(\alpha)(\beta - \alpha)^2$
$= f(\alpha) + (\beta - \alpha)f^{(1)}(\alpha) + \frac{1}{2}f^{(2)}(\alpha)(\beta - \alpha)^2 + \frac{1}{2}Q^{(2)}(\alpha)(\beta - \alpha)^3$
$= f(\alpha) + (\beta - \alpha)f^{(1)}(\alpha) + \frac{1}{2}f^{(2)}(\alpha)(\beta - \alpha)^2 + \frac{1}{2 \cdot 3}f^{(3)}(\alpha)(\beta - \alpha) + \frac{1}{2 \cdot 3}Q^{(3)}(\alpha)(\beta - \alpha)^4$
$\vdots$
$= P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n$

5

5.1 a

For the sake of contradiction, assume $f(x)$ has 2 or more fixed points.
Take fixed points a and b
By MVT, $\exists c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{b-a}{b-a} = 1$
This is a contradiction. Thus, $f(x)$ has at most one fixed point.

5.2 b

If t is a fixed point, then $f(t) = t = t + (1 + e^t)^{-1}$
$0 = \frac{1}{1+e^t}$
No such t exists

5.3 c

For the sake of contradiction, assume no fixed point exists.
$\exists b$ s.t. $f(b) > b$

$\exists a$ s.t. $f(a) < a$ (because $A < 1$, it is impossible for $f(x) > x, \forall x$)
By MVT, $\exists c \in (a, b)$ s.t. $f'(c) = |\frac{f(b)-f(a)}{b-a}| > 1$
Contradiction

$|x_n - x| = |f(x_{n-1} - f(x)|$
By MVT, $existsy \in (x_{n-1}, x$ s.t. $f'(y) = \frac{f(x_{n-1})-f(x)}{x_{n-1}-x}$
$|f(x_{n-1}) - f(x)| = |f'(y)||x_{n-1} - x| \leq A|x_{n-1} - x| = A|f(x_{n-2}) - f(x))| \leq A^{n-1}|x_1 - x|$
As $n \to \infty$, $A^{n-1}|x_1 - x| \to 0$
Thus, $x = \lim x_n$